

# EXPLICIT GROSS-ZAGIER AND WALDSPURGER FORMULAE

LI CAI, JIE SHU, AND YE TIAN

## CONTENTS

1. Main Results	1
1.1. Introduction	1
1.2. The Explicit Gross-Zagier Formula	4
1.3. The Explicit Waldspurger Formula	8
2. Reduction to Local Theory	13
2.1. Petersson Pairing Formula	14
2.2. $U$ -level Pairing	15
2.3. $c_1$ -level Periods	17
2.4. Proof of Main Results	19
3. Local Theory	20
3.1. Local Toric Integrals	21
3.2. Local Orders of Quaternions	22
3.3. Test Vector Spaces	24
3.4. Local Computations	26
References	31

## 1. MAIN RESULTS

**1.1. Introduction.** The Gross-Zagier formula and the Waldspurger formula are probably the two most important analytic tools known at present for studying the still largely unproven conjecture of Birch and Swinnerton-Dyer. Much work has already been done on both formulae. In particular, the recent book by Yuan-Zhang-Zhang [40] establishes what is probably the most general case of the Gross-Zagier formula. Nevertheless, when it comes to actual applications to the arithmetic of elliptic curves or abelian varieties, one very often needs a more explicit form of the Gross-Zagier formula than that given in [40], and similarly a more explicit form of the Waldspurger formula than one finds in the existing literature. This is clearly illustrated, for example, by the papers [1], [33], [34], [9]. Our aim here is to establish what we believe are the most general explicit versions of both formulae, namely Theorems 1.5 and 1.6 for the Gross-Zagier formula, and Theorems 1.8 and 1.9 for the Waldspurger formula. Our methods have been directly inspired by [40], and also the ideas of Gross [14] and Gross-Prasad [15].

In the remainder of this introduction, we would like to explain in detail our explicit formulae in the simplest, and most important, case of modular forms over  $\mathbb{Q}$ . Let  $\phi$  be a newform of weight 2, level  $\Gamma_0(N)$ , with Fourier expansion  $\phi = \sum_{n=1}^{\infty} a_n q^n$  normalized such that  $a_1 = 1$ . Let  $K$  be an imaginary quadratic field of discriminant  $D$  and  $\chi$  a primitive ring class character over  $K$  of conductor  $c$ , i.e. a character of  $\text{Pic}(\mathcal{O}_c)$  where  $\mathcal{O}_c$  is the order  $\mathbb{Z} + c\mathcal{O}_K$  of  $K$ . Assume the Heegner condition (first introduced by Birch in a special case):-

- (1)  $(c, N) = 1$ , and no prime divisor  $p$  of  $N$  is inert in  $K$ , and also  $p$  must be split in  $K$  if  $p^2 | N$ .
- (2)  $\chi([\mathfrak{p}]) \neq a_p$  for any prime  $p | (N, D)$ , where  $\mathfrak{p}$  is the unique prime ideal of  $\mathcal{O}_K$  above  $p$  and  $[\mathfrak{p}]$  is its class in  $\text{Pic}(\mathcal{O}_c)$ .

Let  $L(s, \phi, \chi)$  be the Rankin L-series of  $\phi$  and the theta series  $\phi_\chi$  associated to  $\chi$  (without the local Euler factor at infinity). It follows from the Heegner condition that the sign in the functional equation of  $L(s, \phi, \chi)$  is  $-1$ . Let  $(\phi, \phi)_{\Gamma_0(N)}$  denote the Petersson norm of  $\phi$ :

$$(\phi, \phi)_{\Gamma_0(N)} = \iint_{\Gamma_0(N) \backslash \mathcal{H}} |\phi(z)|^2 dx dy, \quad z = x + iy.$$

---

Ye Tian was supported by grants NSFC 11325106 and NSFC 11321101.

Let  $X_0(N)$  be the modular curve over  $\mathbb{Q}$ , whose  $\mathbb{C}$ -points parametrize isogenies  $E_1 \rightarrow E_2$  between elliptic curves over  $\mathbb{C}$  whose kernel is cyclic of order  $N$ . By the Heegner condition, there exists a proper ideal  $\mathcal{N}$  of  $\mathcal{O}_c$  such that  $\mathcal{O}_c/\mathcal{N} \cong \mathbb{Z}/N\mathbb{Z}$ . For any proper ideal  $\mathfrak{a}$  of  $\mathcal{O}_c$ , let  $P_{\mathfrak{a}} \in X_0(N)$  be the point representing the isogeny  $\mathbb{C}/\mathfrak{a} \rightarrow \mathbb{C}/\mathfrak{a}\mathcal{N}^{-1}$ , which is defined over the ring class field  $H_c$  over  $K$  of conductor  $c$ , and only depends on the class of  $\mathfrak{a}$  in  $\text{Pic}(\mathcal{O}_c)$ . Let  $J_0(N)$  be the Jacobian of  $X_0(N)$ . Writing  $\infty$  for the cusp at infinity on  $X_0(N)$ , we have the morphism from  $X_0(N)$  to  $J_0(N)$  over  $\mathbb{Q}$  given by  $P \mapsto [P - \infty]$ . Let  $P_{\chi}$  be the point

$$P_{\chi} = \sum_{[\mathfrak{a}] \in \text{Pic}(\mathcal{O}_c)} [P_{\mathfrak{a}} - \infty] \otimes \chi([\mathfrak{a}]) \in J_0(N)(H_c) \otimes_{\mathbb{Z}} \mathbb{C},$$

and write  $P_{\chi}^{\phi}$  for the  $\phi$ -isotypical component of  $P_{\chi}$ .

The following theorem was proved by Gross-Zagier in the case  $c = 1$  in their celebrated work [16], and follows immediately from the general explicit Gross-Zagier formula in Theorem 1.5 (see the special case 2, and the example after Theorem 1.5.)

**Theorem 1.1.** *Let  $\phi, \chi$  be as above satisfying the Heegner conditions (1) and (2). Then*

$$L'(1, \phi, \chi) = 2^{-\mu(N, D)} \cdot \frac{8\pi^2(\phi, \phi)_{\Gamma_0(N)}}{u^2 \sqrt{|Dc^2|}} \cdot \widehat{h}_K(P_{\chi}^{\phi}),$$

where  $\mu(N, D)$  is the number of prime factors of the greatest common divisor of  $N$  and  $D$ ,  $u = [\mathcal{O}_c^{\times} : \mathbb{Z}^{\times}]$  is half of the number of roots of unity in  $\mathcal{O}_c$ , and  $\widehat{h}_K$  is the Néron-Tate height on  $J_0(N)$  over  $K$ . In particular, if  $\phi$  is associated to an elliptic curve  $E$  over  $\mathbb{Q}$  via Eichler-Shimura theory and  $f : X_0(N) \rightarrow E$  is a modular parametrization mapping the cusp  $\infty$  to the identity  $O \in E$ , then the Heegner divisor  $P_{\chi}^0(f) := \sum_{[\mathfrak{a}] \in \text{Pic}(\mathcal{O}_c)} f(P_{\mathfrak{a}}) \otimes \chi([\mathfrak{a}]) \in E(H_c)_{\mathbb{C}}$  satisfies

$$L'(1, E, \chi) = 2^{-\mu(N, D)} \cdot \frac{8\pi^2(\phi, \phi)_{\Gamma_0(N)}}{u^2 \sqrt{|Dc^2|}} \cdot \frac{\widehat{h}_K(P_{\chi}^0(f))}{\deg f},$$

where  $\widehat{h}_K$  is the Néron-Tate height on  $E$  over  $K$  and  $\deg f$  is the degree of the morphism  $f$ .

Comparing the above Gross-Zagier formula with the conjecture of Birch and Swinnerton-Dyer for  $L(E/K, s)$ , we immediately obtain the following:

**Conjecture.** *Let  $E$  be an elliptic curve defined over  $\mathbb{Q}$  of conductor  $N$ , and let  $K$  an imaginary quadratic field of discriminant  $D$  such that for any prime  $\ell$  dividing  $N$ , either  $\ell$  splits in  $K$ , or  $\ell$  is ramified in  $K$  and  $\ell^2$  exactly divides  $N$ . Let  $f : X_0(N) \rightarrow E$  be a modular parametrization mapping  $\infty$  to  $O$ . Let  $\mathcal{N} \subset \mathcal{O}_K$  be any ideal with  $\mathcal{O}_K/\mathcal{N} \cong \mathbb{Z}/N\mathbb{Z}$ , let  $P \in X_0(N)(H_K)$  be the point representing the isogeny  $(\mathbb{C}/\mathcal{O}_K \rightarrow \mathbb{C}/\mathcal{N}^{-1})$ , and write  $P_K(f) := \text{Tr}_{H_K/K} f(P) \in E(K)$ . Assume  $P_K(f)$  is not torsion. Then*

$$\sqrt{\#\text{III}(E/K)} = 2^{-\mu(N, D)} \cdot \frac{[E(K) : \mathbb{Z}P_K(f)]}{C \cdot [\mathcal{O}_K^{\times} : \mathbb{Z}^{\times}] \cdot \prod_{\ell | \frac{N}{(N, D)}} m_{\ell}},$$

where  $m_{\ell} = [E(\mathbb{Q}_{\ell}) : E^0(\mathbb{Q}_{\ell})]$ , and  $C$  is the positive integer such that if  $\omega_0$  is a Néron differential on  $E$  then  $f^*\omega_0 = \pm C \cdot 2\pi i \phi(z) dz$ .

We next state our explicit Waldspurger formula over  $\mathbb{Q}$ . Let  $\phi = \sum_{n=1}^{\infty} a_n q^n \in S_2(\Gamma_0(N))$  be a newform of weight 2 and level  $\Gamma_0(N)$ . Let  $K$  be an imaginary quadratic field and  $\chi : \text{Gal}(H_c/K) \rightarrow \mathbb{C}^{\times}$  a character of conductor  $c$ . Assume the following conditions:

- (i)  $(c, N) = 1$  and if  $p | (N, D)$ , then  $p^2 \nmid N$ ;
- (ii) let  $S$  be the set of places  $p | N\infty$  non-split in  $K$  such that for a finite prime  $p$ ,  $\text{ord}_p(N)$  is odd if  $p$  is inert in  $K$ , and  $\chi([\mathfrak{p}]) = a_p$  if  $p$  is ramified in  $K$ . Then  $S$  has even cardinality.

It follows that the sign of the functional equation of the Rankin L-series  $L(s, \phi, \chi)$  is  $+1$ . Let  $B$  be the quaternion algebra over  $\mathbb{Q}$  ramified exactly at places in  $S$ . Note that the condition (ii) implies that there exists an embedding of  $K$  into  $B$  which we fix once and for all. Let  $R \subset B$  be an order of discriminant  $N$  with  $R \cap K = \mathcal{O}_c$ . Such an order exists and is unique up to conjugation by  $\widehat{K}^{\times}$ . Here, for an abelian group  $M$ , we define  $\widehat{M} = M \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ , where  $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ , with  $p$  running over all primes. By the reduction theory of definite quadratic forms, the coset  $X := B^{\times} \backslash \widehat{B}^{\times} / \widehat{R}^{\times}$  is finite, say of order  $n$ . Let  $g_1, \dots, g_n$  in  $\widehat{B}^{\times}$  represent the distinct classes  $[g_1], \dots, [g_n]$ . For each  $i = 1, \dots, n$ , let  $\Gamma_i = (B^{\times} \cap g_i \widehat{B}^{\times} g_i^{-1}) / \{\pm 1\}$ .

Then  $\Gamma_i$  is a finite group, and we denote its order by  $w_i$ . Let  $\mathbb{Z}[X]$  denote the free  $\mathbb{Z}$ -module of formal sums  $\sum_{i=1}^n a_i [g_i]$  with  $a_i \in \mathbb{Z}$ , and define a height pairing on  $\mathbb{Z}[X]$  by

$$\langle \sum a_i [g_i], \sum b_i [g_i] \rangle = \sum_{i=1}^n a_i b_i w_i,$$

which is positive definite on  $\mathbb{R}[X] := \mathbb{Z}[X] \otimes_{\mathbb{Z}} \mathbb{R}$  and has a natural Hermitian extension to  $\mathbb{C}[X] := \mathbb{Z}[X] \otimes_{\mathbb{Z}} \mathbb{C}$ . Define the degree of a vector  $\sum a_i [g_i] \in \mathbb{Z}[X]$  to be  $\sum a_i$  and let  $\mathbb{Z}[X]^0$  denote the degree 0 submodule of  $\mathbb{Z}[X]$ . Then  $\mathbb{Z}[X]$  and  $\mathbb{Z}[X]^0$  are endowed with actions of Hecke operators  $T_p, S_p, p \nmid N$  which are linear and defined as follows. For any prime  $p \nmid N$ ,  $B_p^\times / R_p^\times \cong \mathrm{GL}_2(\mathbb{Q}_p) / \mathrm{GL}_2(\mathbb{Z}_p)$  can be identified with the set of  $\mathbb{Z}_p$ -lattices in a 2-dimensional vector space over  $\mathbb{Q}_p$ . Then for any  $g = (g_v) \in \widehat{B}^\times$ ,

$$S_p([g]) = [g^{(p)} s_p(g_p)], \quad T_p([g]) = \sum_{h_p} [g^{(p)} h_p],$$

where  $g^{(p)}$  is the  $p$ -off part of  $g$ , namely  $g^{(p)} = (g_v^{(p)})$  with  $g_v^{(p)} = g_v$  for all  $v \neq p$  and  $g_p^{(p)} = 1$ , and if  $g_p$  corresponds to lattice  $\Lambda$ , then  $s_p(g_p)$  is the coset corresponding to the homothetic lattice  $p\Lambda$ , and  $h_p$  runs over  $p+1$  lattices  $\Lambda' \subset \Lambda$  with  $[\Lambda : \Lambda'] = p$ . There is a unique line  $V_\phi \subset \mathbb{C}[X]^0$  where  $T_p$  acts as  $a_p$  and  $S_p$  acts trivially for all  $p \nmid N$ . Recall that the fixed embedding of  $K$  into  $B$  induces a map

$$\mathrm{Pic}(\mathcal{O}_c) = K^\times \backslash \widehat{K}^\times / \widehat{\mathcal{O}_c}^\times \longrightarrow X = B^\times \backslash \widehat{B}^\times / \widehat{R}^\times, \quad t \mapsto x_t,$$

using which we define an element in  $\mathbb{C}[X]$ ,

$$P_\chi := \sum \chi^{-1}(t) x_t$$

and let  $P_\chi^\phi$  be its projection to the line  $V_\phi$ . The following explicit height formula for  $P_\chi^\phi$ , which was proved by Gross in some case in [13], is a special case of the explicit Waldspurger formulas in Theorems 1.8 and 1.10 (with Proposition 3.8).

**Theorem 1.2.** *Let  $(\phi, \chi)$  be as above satisfying the conditions (i) and (ii). Then we have*

$$L(1, \phi, \chi) = 2^{-\mu(N, D)} \cdot \frac{8\pi^2(\phi, \phi)_{\Gamma_0(N)}}{u^2 \sqrt{|Dc^2|}} \cdot \langle P_\chi^\phi, P_\chi^\phi \rangle,$$

where  $\mu(N, D)$  is the number of prime factors of the greatest common divisor of  $N$  and  $D$ ,  $u = [\mathcal{O}_c^\times : \mathbb{Z}^\times]$  is half of number of roots of unity in  $\mathcal{O}_c$ . Let  $f = \sum_i f(g_i) w_i^{-1} [g_i]$  be any non-zero vector on the line  $V_\phi$  and let  $P_\chi^0(f) = \sum_{t \in \mathrm{Pic}(\mathcal{O}_c)} f(t) \chi(t)$ . Then the above formula can be rewritten as

$$L(1, \phi, \chi) = 2^{-\mu(N, D)} \cdot \frac{8\pi^2(\phi, \phi)_{\Gamma_0(N)}}{u^2 \sqrt{|Dc^2|}} \cdot \frac{|P_\chi^0(f)|^2}{\langle f, f \rangle}.$$

**Acknowledgements.** The authors thank J. Coates, H. Darmon, B. Gross, D. Prasad, W. Xiong, X. Yuan, S. Zhang, and W. Zhang for encouragement and helpful discussions.

**Notations for First Two Sections.** We denote by  $F$  the base number field of degree  $d = [F : \mathbb{Q}]$  over  $\mathbb{Q}$  and  $\mathcal{O} = \mathcal{O}_F$  its ring of integers with different  $\delta$ . Let  $\mathbb{A} = F_\mathbb{A}$  be the adèle ring of  $F$  and  $\mathbb{A}_f$  its finite part. For any  $\mathbb{Z}$ -module  $M$ , we denote by  $\widehat{M} = M \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$  and  $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ . For example,  $\widehat{F} = \mathbb{A}_f$ . Let  $|\cdot|_\mathbb{A} : \mathbb{A}^\times \longrightarrow \mathbb{R}_+^\times$  denote the standard adèlic absolute value so that  $d(ab) = |a|_\mathbb{A} db$  for any Haar measure  $db$  on  $\mathbb{A}$ . Let  $|\cdot|_v$  denote the absolute value on  $F_v^\times$  for each place  $v$  of  $F$  such that  $|x|_\mathbb{A} = \prod_v |x_v|_v$  for any  $x = (x_v) \in \mathbb{A}^\times$ . For any non-zero fractional ideal  $b$  of  $F$ , let  $\|b\|$  denote the norm of  $b$ . For any  $x \in \mathbb{A}_f^\times$  we also write  $\|x\|$  for  $\|b_x\|$  where  $b_x$  is the ideal corresponding to  $x$  so that  $\|x\| = |x|_\mathbb{A}^{-1}$  and for any non-zero fractional ideal  $b$  we also write  $|b|_\mathbb{A}$  for  $|x_b|_\mathbb{A}$  with any  $x_b \in \mathbb{A}_f^\times$  whose corresponding ideal is  $b$  so that  $|b|_\mathbb{A} = \|\mathbb{A}\|^{-1}$ . For a finite place  $v$ , sometimes we also denote by  $v$  its corresponding prime ideal and  $q_v = \#\mathcal{O}/v$ . For a fractional ideal  $b$  of  $F$ , we write  $|b|_v = |x_b|_v$  for  $x_b \in F_v$  with  $x_b \mathcal{O}_v = b \mathcal{O}_v$ , denote by  $\mathrm{ord}_v(b)$  the additive valuation of  $b$  at  $v$  so that  $\mathrm{ord}_v(v) = 1$  and write  $v\|b\|$  if  $\mathrm{ord}_v(b) = 1$ . We denote by  $\infty$  the set of infinite places of  $F$ . Denote by  $L(s, 1_F)$  the complete L-series for the trivial Hecke character  $1_F$  on  $\mathbb{A}^\times$  so that  $L(s, 1_F) = \Gamma_\mathbb{R}(s)^{r_1} \Gamma_\mathbb{C}(s)^{r_2} \zeta_F(s)$ , where  $r_1$  (resp  $r_2$ ) is the number of real (resp. complex) places of  $F$ ,  $\zeta_F(s)$  is the usual Dedekind zeta-function of  $F$ ,  $\Gamma_\mathbb{R}(s) = \pi^{-s/2} \Gamma(s/2)$ , and  $\Gamma_\mathbb{C}(s) = 2(2\pi)^{-s} \Gamma(s)$ . For each place  $v$  of  $F$ , let  $L(s, 1_v)$  denote the local Euler factor of  $L(s, 1_F)$  at  $v$ . Let  $D_F$  denote the absolute discriminant of  $F$  and  $\delta \subset \mathcal{O}$  the different of  $F$  so that  $\|\delta\| = |D_F|$ .

In the first two sections, we denote by  $K$  a quadratic extension over  $F$ ,  $D = D_{K/F} \subset \mathcal{O}$  the relative discriminant of  $K$  over  $F$ , and  $D_K$  the absolute discriminant of  $K$ . Let  $K^{\text{ab}}$  denote the maximal abelian extension over  $K$  and let  $\sigma : K_{\mathbb{A}}^{\times}/K^{\times} \rightarrow \text{Gal}(K^{\text{ab}}/K)$  denote the Artin reciprocity map in the class field theory. For any non-zero ideal  $b$  of  $\mathcal{O}$  let  $\mathcal{O}_b = \mathcal{O} + b\mathcal{O}_K$  be the unique  $\mathcal{O}$ -order of  $K$  satisfying  $[\mathcal{O}_K : \mathcal{O}_b] = \#\mathcal{O}/b$  and we call  $b$  its conductor. For any finite place  $v$  of  $F$ ,  $\mathcal{O}_{b,v} = \mathcal{O}_b \otimes_{\mathcal{O}} \mathcal{O}_v$  only depends on  $\text{ord}_v b$ . Thus for a fractional ideal  $b$  and a finite place  $v$  of  $F$ ,  $\mathcal{O}_{b,v}$  makes sense if  $\text{ord}_v b \geq 0$ . Let  $\text{Pic}_{K/F}(\mathcal{O}_b) = \widehat{K}^{\times}/K^{\times}\widehat{F}^{\times}\widehat{\mathcal{O}}_b^{\times}$ . Then there is an exact sequence

$$\text{Pic}(\mathcal{O}_F) \rightarrow \text{Pic}(\mathcal{O}_b) \rightarrow \text{Pic}_{K/F}(\mathcal{O}_b) \rightarrow 0.$$

Let  $\kappa_b$  be the kernel of the first arrow, which has order 1 or 2 in the case  $F$  is totally real and  $K$  is a totally imaginary quadratic extension over  $F$  (see Theorem 10.3 in [39]).

For any algebraic group  $G$  over  $F$ , let  $G_{\mathbb{A}} = G(\mathbb{A})$  be the group of adélic points on  $G$ . For a finite set  $S$  of places of  $F$ , let  $G_S = \prod_{v \in S} G(F_v)$  (resp.  $G_{\mathbb{A}}^{(S)} = G(\mathbb{A})^{(S)}$ ) the  $S$ -part of  $G_{\mathbb{A}}$  (resp. the  $S$ -off part of  $G_{\mathbb{A}}$ ) viewed as a subgroup of  $G_{\mathbb{A}}$  naturally so that the  $S$ -off components (resp,  $S$ -components) are constant 1. More general, for a subgroup  $U$  of  $G_{\mathbb{A}}$  of form  $U = U_T U^T$  for some set  $T$  of places disjoint with  $S$  where  $U_T \subset \prod_{v \in T} G(F_v)$  and  $U^T = \prod_{v \notin T} U_v$  with  $U_v$  a subgroup of  $G(F_v)$ , we may define  $U^{(S)}$ ,  $U_S$ , and view them as subgroups of  $U$  similarly. For any ideal  $b$  of  $\mathcal{O}$ , we also write  $U^{(b)}$  for  $U^{(S_b)}$ , and  $U_b$  for  $U_{S_b}$ , with  $S_b$  the set of places dividing  $b$ . Let  $U_0(N)$  and  $U_1(N)$  denote subgroups of  $\text{GL}_2(\widehat{\mathcal{O}})$  defined by

$$U_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\widehat{\mathcal{O}}) \mid c \in N\widehat{\mathcal{O}} \right\}, \quad U_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_0(N) \mid d \equiv 1 \pmod{N\widehat{\mathcal{O}}} \right\}.$$

When  $F$  is totally real field and  $\sigma$  is an automorphic cuspidal representation of level  $N$  such that  $\sigma_v$  is a discrete series for all  $v|\infty$ , for an automorphic form  $\phi$  of level  $U_1(N)$ , we let  $(\phi, \phi)_{U_0(N)}$  denote the Petersson norm defined using the invariant measure  $dx dy/y^2$  on the upper half plane.

**1.2. The Explicit Gross-Zagier Formula.** Let  $F$  be a totally real number field of degree  $d$ ,  $\mathbb{A} = \mathbb{A}_F$  the adéle ring of  $F$ , and  $\mathbb{A}_f$  its finite part. Let  $\mathbb{B}$  be an incoherent quaternion algebra over  $\mathbb{A}$ , totally definite at infinity. For each open compact subgroup  $U$  of  $\mathbb{B}_f^{\times} = (\mathbb{B} \otimes_{\mathbb{A}} \mathbb{A}_f)^{\times}$ , let  $X_U$  be the Shimura curve over  $F$  associated to  $U$  and  $\xi_U \in \text{Pic}(X_U)_{\mathbb{Q}}$  the normalized Hodge class on  $X_U$ , i.e. the unique line bundle, which has degree one on each geometrically connected components, and is parallel to

$$\omega_{X_U/F} + \sum_{x \in X_U(\overline{F})} (1 - e_x^{-1})x.$$

Here  $\omega_{X_U/F}$  is the canonical bundle of  $X_U$ ,  $e_x$  is the ramification index of  $x$  in the complex uniformization of  $X_U$ , i.e. for a cusp  $x$ ,  $e_x = \infty$  so that  $1 - e_x^{-1} = 1$ ; for a non-cusp  $x$ ,  $e_x$  is the ramification index of any preimage of  $x$  in the map  $X_{U'} \rightarrow X_U$  for any sufficiently small open compact subgroup  $U'$  of  $U$  such that each geometrically connected component of  $X_{U'}$  is a free quotient of  $\mathcal{H}$  under the complex uniformization. For any two open compact subgroups  $U_1 \subset U_2$  of  $\mathbb{B}_f^{\times}$ , there is a natural surjective morphism  $X_{U_1} \rightarrow X_{U_2}$ . Let  $X$  be the projective limit of the system  $(X_U)_U$ , which is endowed with the Hecke action of  $\mathbb{B}^{\times}$  where  $\mathbb{B}_{\infty}^{\times}$  acts trivially. Note that each  $X_U$  is the quotient of  $X$  by the action of  $U$ .

Let  $A$  be a simple abelian variety over  $F$  parametrized by  $X$  in the sense that there is a non-constant morphism  $X_U \rightarrow A$  over  $F$  for some  $U$ . Then by Eichler-Shimura theory,  $A$  is of strict  $\text{GL}(2)$ -type in the sense that  $M := \text{End}^0(A) = \text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$  is a field and  $\text{Lie}(A)$  is a free module of rank one over  $M \otimes_{\mathbb{Q}} F$  by the induced action. Let

$$\pi_A = \text{Hom}_{\xi}^0(X, A) := \varinjlim_U \text{Hom}_{\xi_U}^0(X_U, A),$$

where  $\text{Hom}_{\xi_U}^0(X_U, A)$  denotes the morphisms in  $\text{Hom}(X_U, A) \otimes_{\mathbb{Z}} \mathbb{Q}$  using  $\xi_U$  as a base point: if  $\xi_U$  is represented by a divisor  $\sum_i a_i x_i$  on  $X_{U, \overline{F}}$ , then  $f \in \text{Hom}_F(X_U, A) \otimes_{\mathbb{Z}} \mathbb{Q}$  is in  $\pi_A$  if and only if  $\sum_i a_i f(x_i) = 0$  in  $A(\overline{F})_{\mathbb{Q}} := A(\overline{F}) \otimes_{\mathbb{Z}} \mathbb{Q}$ . For each open compact subgroup  $U$  of  $\mathbb{B}_f^{\times}$ , let  $J_U$  denote the Jacobian of  $X_U$ . Then  $\pi_A = \text{Hom}^0(J, A) := \varinjlim_U \text{Hom}^0(J_U, A)$  where  $\text{Hom}^0(J_U, A) = \text{Hom}_F(J_U, A) \otimes_{\mathbb{Z}} \mathbb{Q}$ . The action of  $\mathbb{B}^{\times}$  on  $X$  induces a natural  $\mathbb{B}^{\times}$ -module structure on  $\pi_A$  so that  $\text{End}_{\mathbb{B}^{\times}}(\pi_A) = M$  and has a decomposition  $\pi_A = \otimes_M \pi_{A,v}$  where  $\pi_{A,v}$  are absolutely irreducible representations of  $\mathbb{B}_v^{\times}$  over  $M$ . Using Jacquet-Langlands correspondence, one can define the complete L-series of  $\pi_A$

$$L(s, \pi_A) = \prod_v L(s, \pi_{A,v}) \in M \otimes_{\mathbb{Q}} \mathbb{C}$$

as an entire function of  $s \in \mathbb{C}$ . Let  $L(s, A, M)$  denote the L-series of  $\ell$ -adic Galois representation with coefficients in  $M \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell}$  associated to  $A$  (without local Euler factors at infinity), then  $L_v(s, A, M) = L(s - \frac{1}{2}, \pi_v)$  for all finite places  $v$  of  $F$ . Let  $A^{\vee}$  denote the dual abelian variety of  $A$ . There is perfect  $\mathbb{B}^{\times}$ -invariant pairing

$$\pi_A \times \pi_{A^{\vee}} \longrightarrow M$$

given by

$$(f_1, f_2) = \text{Vol}(X_U)^{-1} (f_{1,U} \circ f_{2,U}^{\vee}), \quad f_{1,U} \in \text{Hom}(J_U, A), \quad f_{2,U} \in \text{Hom}(J_U, A^{\vee}),$$

where  $f_{2,U}^{\vee} : A \rightarrow J_U$  is the dual of  $f_{2,U}$  composed with the canonical isomorphism  $J_U^{\vee} \simeq J_U$ . Here  $\text{Vol}(X_U)$  is defined by a fixed invariant measure on the upper half plane. It follows that  $\pi_{A^{\vee}}$  is dual to  $\pi_A$  as representations of  $\mathbb{B}^{\times}$  over  $M$ . For any fixed open compact subgroup  $U$  of  $\mathbb{B}_f^{\times}$ , define the  $U$ -pairing on  $\pi_A \times \pi_{A^{\vee}}$  by

$$(f_1, f_2)_U = \text{Vol}(X_U) (f_1, f_2), \quad f_1 \in \pi_A, f_2 \in \pi_{A^{\vee}}$$

which is independent of the choice of measure defining  $\text{Vol}(X_U)$ . When  $A$  is an elliptic curve and identify  $A^{\vee}$  with  $A$  canonically, then for any morphism  $f : X_U \rightarrow A$ , we have  $(f, f)_U = \deg f$ , the degree of the finite morphism  $f$ .

Let  $K$  be a totally imaginary quadratic extension over  $F$  with associated quadratic character  $\eta$  on  $\mathbb{A}^{\times}$ . Let  $L$  be a finite extension of  $M$  and  $\chi : K^{\times} \backslash K_{\mathbb{A}}^{\times} \rightarrow L^{\times}$  an  $L$ -valued Hecke character of finite order. Let  $L(s, A, \chi) \in L \otimes_{\mathbb{Q}} \mathbb{C}$  be the complete L-series obtained by  $\ell$ -adic Galois representation associated to  $A$  tensored with the induced representation of  $\chi$  from  $\text{Gal}(\overline{K}/K)$  to  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ . Assume that

$$\omega_A \cdot \chi|_{\mathbb{A}^{\times}} = 1,$$

where  $\omega_A$  is the central character of  $\pi_A$  on  $\mathbb{A}_f^{\times}$ , and that for each finite place  $v$  of  $F$ ,

$$\epsilon(\pi_{A,v}, \chi_v) = \chi_v \eta_v(-1) \epsilon(\mathbb{B}_v),$$

where  $\epsilon(\mathbb{B}_v) = 1$  if  $\mathbb{B}_v$  is split and  $= -1$  otherwise, and  $\epsilon(\pi_{A,v}, \chi_v) = \epsilon(1/2, \pi_{A,v}, \chi_v)$  is the local root number of  $L(s, \pi_A, \chi)$ . It follows that the global root number of the L-series  $L(s, \pi_A, \chi)$  is  $-1$  and there is an embedding of  $K_{\mathbb{A}}$  into  $\mathbb{B}$  over  $\mathbb{A}$ . We fixed such an embedding once for all and then view  $K_{\mathbb{A}}^{\times}$  as a subgroup of  $\mathbb{B}^{\times}$ .

Let  $N$  be the conductor of  $\pi^{\text{JL}}$ ,  $D$  the relative discriminant of  $K$  over  $F$ ,  $c \subset \mathcal{O}$  be the ideal maximal such that  $\chi$  is trivial on  $\prod_{v \nmid c} \mathcal{O}_{K_v}^{\times} \prod_{v|c} (1 + c\mathcal{O}_{K,v})$ . Define the following sets of places  $v$  of  $F$  dividing  $N$ :

$$\Sigma_1 := \{v|N \text{ nonsplit in } K : \text{ord}_v(c) < \text{ord}_v(N)\}.$$

Let  $c_1 = \prod_{\mathfrak{p}|c, \mathfrak{p} \notin \Sigma_1} \mathfrak{p}^{\text{ord}_{\mathfrak{p}} c}$  be the  $\Sigma_1$ -off part of  $c$ ,  $N_1$  the  $\Sigma_1$ -off part of  $N$ , and  $N_2 = N/N_1$ .

Let  $v$  be a place of  $F$  and  $\varpi_v$  a uniformizer of  $F_v$ . Then there exists an  $\mathcal{O}_v$ -order  $R_v$  of  $\mathbb{B}_v$  with discriminant  $N\mathcal{O}_v$  such that  $R_v \cap K_v = \mathcal{O}_{c_1,v}$ . Such an order  $R_v$  is called admissible for  $(\pi_v, \chi_v)$  if it also satisfies the following conditions (1) and (2). Note that up to  $K_v^{\times}$ -conjugate there is a unique such order when  $v \nmid (c_1, N)$ , and that  $\mathbb{B}$  must be split at places  $v|(c_1, N)$  by Lemma 3.1.

- (1) If  $v|(c_1, N)$ , then  $R_v$  is the intersection of two maximal orders  $R'_v, R''_v$  of  $\mathbb{B}_v$  such that  $R'_v \cap K_v = \mathcal{O}_{c,v}$  and

$$R''_v \cap K_v = \begin{cases} \mathcal{O}_{c/N,v}, & \text{if } \text{ord}_v(c/N) \geq 0, \\ \mathcal{O}_{K,v}, & \text{otherwise.} \end{cases}$$

Note that for  $v|(c_1, N)$ , there is a unique order, up to  $K_v^{\times}$ -conjugate, satisfying the condition (1) unless  $\text{ord}_v(c_1) < \text{ord}_v(N)$ . In the case  $0 < \text{ord}_v(c_1) < \text{ord}_v(N)$ ,  $v$  must split in  $K$  by the definition of  $\Sigma_1$  and there are exactly two  $K_v^{\times}$ -conjugacy classes of orders satisfying the condition (1), which are conjugate to each other by a normalizer of  $K_v^{\times}$  in  $\mathbb{B}_v^{\times}$ . Fix an  $F_v$ -algebra isomorphism  $K_v \cong F_v^2$  and identify  $\mathbb{B}_v$  with  $\text{End}_{F_v}(K_v)$ . Then the two classes contain respectively orders  $R_{i,v} = R'_{i,v} \cap R''_{i,v}$ ,  $i = 1, 2$  as in (1) such that  $R'_{i,v} = \text{End}_{\mathcal{O}}(\mathcal{O}_c)$ ,  $i = 1, 2$ , and  $R''_{1,v} = \text{End}_{\mathcal{O}_v}((\varpi_v^{n-c}, 1)\mathcal{O}_{K_v})$  and  $R''_{2,v} = \text{End}_{\mathcal{O}_v}((1, \varpi_v^{n-c})\mathcal{O}_{K_v})$ .

- (2) If  $0 < \text{ord}_v(c_1) < \text{ord}_v(N)$ , then  $R_v$  is  $K_v^{\times}$ -conjugate to some  $R_{i,v}$  such that  $\chi_i$  has conductor  $\text{ord}_v(c)$ , where  $\chi_i$ ,  $i = 1, 2$  is defined by  $\chi_1(a) = \chi_v(a, 1)$  and  $\chi_2(b) = \chi_v(1, b)$ .

**Definition 1.3.** An  $\widehat{\mathcal{O}}$ -order  $\mathcal{R}$  of  $\mathbb{B}_f$  is called admissible for  $(\pi, \chi)$  if for every finite place  $v$  of  $F$ ,  $\mathcal{R}_v := \mathcal{R} \otimes_{\widehat{\mathcal{O}}} \mathcal{O}_v$  is admissible for  $(\pi_v, \chi_v)$ . Note that an admissible order  $\mathcal{R}$  for  $(\pi, \chi)$  is of discriminant  $N\widehat{\mathcal{O}}$  such that  $\mathcal{R} \cap \widehat{K} = \widehat{\mathcal{O}}_{c_1}$ .



Let  $\mathcal{R}$  be an  $\widehat{\mathcal{O}}$ -order of  $\mathbb{B}_f$  with discriminant  $N$  such that  $\mathcal{R} \cap K_{\mathbb{A}_f} = \widehat{\mathcal{O}}_{c_1}$  and that  $\mathcal{R}_v := \mathcal{R} \otimes_{\widehat{\mathcal{O}}} \mathcal{O}_v$  is admissible with  $\chi_v$  for any places  $v$  in the above sense. Note that  $\mathcal{R}_v$  is unique up to  $K_v^\times$ -conjugate for any  $v \nmid (c_1, N)$ .

Let  $U = \mathcal{R}^\times$  and  $U^{(N_2)} := \mathcal{R}^\times \cap \mathbb{B}_f^{\times(N_2)}$ . Note that for any finite place  $v|N_1$ ,  $\mathbb{B}_v$  must be split (by Lemma 3.1 (5)). Let  $Z \cong \mathbb{A}_f^\times$  denote the center of  $\mathbb{B}_f^\times$ . The group  $U^{(N_2)}$  has a decomposition  $U^{(N_2)} = U' \cdot (Z \cap U^{(N_2)})$  where  $U' = \prod_{v \nmid N_2 \infty} U'_v$  such that for any finite place  $v \nmid N_2$ ,  $U'_v = U_v$  if  $v \nmid N$  and  $U'_v \cong U_1(N)_v$  otherwise. View  $\omega$  as a character on  $Z$  and we may define a character on  $U^{(N_2)}$  by  $\omega$  on  $Z \cap U^{(N_2)}$  and trivial on  $U'$ , which we also denoted by  $\omega$ .

**Definition 1.4.** Let  $V(\pi, \chi)$  denote the space of forms  $f \in \pi_A \otimes_M L$ , which are  $\omega$ -eigen under  $U^{(N_2)}$ , and  $\chi_v^{-1}$ -eigen under  $K_v^\times$  for all places  $v \in \Sigma_1$ . The space  $V(\pi, \chi)$  is actually a one dimensional  $L$ -space (see Proposition 3.7).

Consider the Hecke action of  $K_{\mathbb{A}}^\times \subset \mathbb{B}^\times$  on  $X$ . Let  $X^{K^\times}$  be the  $F$ -subscheme of  $X$  of fixed points of  $X$  under  $K^\times$ . The theory of complex multiplication asserts that every point in  $X^{K^\times}(\bar{F})$  is defined over  $K^{\text{ab}}$  and that the Galois action is given by the Hecke action under the reciprocity law. Fix a point  $P \in X^{K^\times}$  and let  $f \in V(\pi, \chi)$  be a non-zero vector. Define a Heegner cycle associated to  $(\pi, \chi)$  to be

$$P_\chi^0(f) := \sum_{t \in \text{Pic}_{K/F}(\mathcal{O}_{c_1})} f(P)^{\sigma_t} \chi(t) \in A(K^{\text{ab}})_{\mathbb{Q}} \otimes_M L,$$

where  $\text{Pic}_{K/F}(\mathcal{O}_{c_1}) = \widehat{K}^\times / K^\times \widehat{F}^\times \widehat{\mathcal{O}}_{c_1}^\times$  and  $t \mapsto \sigma_t$  is the reciprocity law map in the class field theory. The Néron-Tate height pairing over  $K$  gives a  $\mathbb{Q}$ -linear map  $\langle \cdot, \cdot \rangle_K : A(\bar{K})_{\mathbb{Q}} \otimes_M A^\vee(\bar{K})_{\mathbb{Q}} \rightarrow \mathbb{R}$ . Let  $\langle \cdot, \cdot \rangle_{K,M} : A(\bar{K})_{\mathbb{Q}} \otimes_M A^\vee(\bar{K})_{\mathbb{Q}} \rightarrow M \otimes_{\mathbb{Q}} \mathbb{R}$  be the unique  $M$ -bilinear pairing such that  $\langle \cdot, \cdot \rangle_K = \text{tr}_{M \otimes \mathbb{R}/\mathbb{R}} \langle \cdot, \cdot \rangle_{K,M}$ . The pairing  $\langle \cdot, \cdot \rangle_{K,M}$  induces an  $L$ -linear Néron-Tate pairing over  $K$ :

$$\langle \cdot, \cdot \rangle_{K,L} : (A(\bar{K})_{\mathbb{Q}} \otimes_M L) \otimes_L (A^\vee(\bar{K})_{\mathbb{Q}} \otimes_M L) \longrightarrow L \otimes_{\mathbb{Q}} \mathbb{R}.$$

Note that the  $\mathbb{B}^\times$ -invariant  $M$ -linear pairing  $(\cdot, \cdot)_U : \pi_A \times \pi_{A^\vee} \rightarrow M$  induces a  $\mathbb{B}^\times$ -invariant  $L$ -linear pairing

$$(\cdot, \cdot)_U : (\pi_A \otimes_M L) \times (\pi_{A^\vee} \otimes_M L) \longrightarrow L.$$

The Hilbert newform  $\phi$  in the Jacquet-Langlands correspondence  $\sigma$  of  $\pi_A$  on  $\text{GL}_2(\mathbb{A})$  is the form of level  $U_1(N)$ , for each  $v|\infty$ ,  $\text{SO}_2(\mathbb{R}) \subset \text{GL}_2(F_v)$  acts by the character  $\sigma(k_\theta)\phi = e^{4\pi i\theta}\phi$  where  $k_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \in \text{SO}_2(\mathbb{R})$ , such that

$$L(s, \pi) = 2^d \cdot |\delta|_{\mathbb{A}}^{s-\frac{1}{2}} \cdot Z(s, \phi), \quad Z(s, \phi) = \int_{F^\times \backslash \mathbb{A}^\times} \phi \begin{pmatrix} a & \\ & 1 \end{pmatrix} |a|_{\mathbb{A}}^{s-\frac{1}{2}} d^\times a$$

where the measure  $d^\times a$  is taken to be Tamagawa measure so that  $\text{Res}_{s=1} \int_{|a| \leq 1, a \in F^\times \backslash \mathbb{A}^\times} |a|^{s-1} d^\times a = \text{Res}_{s=1} L(s, 1_F)$ , and  $\delta$  is the differential of  $F$ . Note that  $\phi(g)\bar{\phi}(g)$  is a function on

$$\text{GL}_2(F)_+ \backslash \text{GL}_2(F_\infty)_+ \times \text{GL}_2(\mathbb{A}_f)/Z(\mathbb{A}) \cdot (U_{1,\infty} \times U_0(N)) \cong \text{GL}_2(F)_+ \backslash \mathcal{H}^d \times \text{GL}_2(\mathbb{A}_f)/U_0(N)\mathbb{A}_f^\times.$$

We define the Petersson norm  $(\phi, \phi)_{U_0(N)}$  by the integration of  $\phi\bar{\phi}$  with measure  $dxdy/y^2$  on each upper half plane. One main result of this paper is the following.

**Theorem 1.5** (Explicit Gross-Zagier Formula). *Let  $F$  be a totally real field of degree  $d$ . Let  $A$  be an abelian variety over  $F$  parametrized by a Shimura curve  $X$  over  $F$  and  $\phi$  the Hilbert holomorphic newform of parallel weight 2 on  $\text{GL}_2(\mathbb{A})$  associated to  $A$ . Let  $K$  be a totally imaginary quadratic extension over  $F$  with relative discriminant  $D$  and discriminant  $D_K$ . Let  $\chi : K_{\mathbb{A}}^\times / K^\times \rightarrow L^\times$  be a finite Hecke character of conductor  $c$  over some finite extension  $L$  of  $M := \text{End}^0(A)$ . Assume that*

- (1)  $\omega_A \cdot \chi|_{\mathbb{A}^\times} = 1$ , where  $\omega_A$  is the central character of  $\pi_A$ ;
- (2) for any place  $v$  of  $F$ ,  $\epsilon(\pi_{A,v}, \chi_v) = \chi_v \eta_v(-1) \epsilon(\mathbb{B}_v)$ .

For any non-zero forms  $f_1 \in V(\pi_A, \chi)$  and  $f_2 \in V(\pi_{A^\vee}, \chi^{-1})$ , we have an equality in  $L \otimes_{\mathbb{Q}} \mathbb{C}$ :

$$L'(\Sigma)(1, A, \chi) = 2^{-\#\Sigma_D} \cdot \frac{(8\pi^2)^d \cdot (\phi, \phi)_{U_0(N)}}{u_1^2 \sqrt{|D_K|} \|c_1^2\|} \cdot \frac{\langle P_\chi^0(f_1), P_{\chi^{-1}}^0(f_2) \rangle_{K,L}}{(f_1, f_2)_{\mathcal{R}^\times}}$$

where

$$\Sigma := \{v|(N, Dc) : \text{if } v|N \text{ then } \text{ord}_v(c/N) \geq 0\},$$

$$\Sigma_D := \{v|(N, D) : \text{ord}_v(c) < \text{ord}_v(N)\},$$

the ideal  $c_1|c$  is the  $\Sigma_1$ -off part of  $c$  as before,  $u_1 = \#\kappa_{c_1} \cdot [\mathcal{O}_{c_1}^\times : \mathcal{O}^\times]$  and  $\kappa_{c_1}$  is the kernel of the morphism from  $\text{Pic}(\mathcal{O})$  to  $\text{Pic}(\mathcal{O}_{c_1})$  which has order 1 or 2, and  $(\phi, \phi)_{U_0(N)}$  is the Petersson norm with respect to the measure  $dx dy/y^2$  on the upper half plane.

**Remark:** Note that the assumption  $\omega_A|_{\mathbb{A}^\times} \cdot \chi = 1$  implies  $L(s, A, \chi) = L(s, A^\vee, \chi^{-1})$ . Let  $\phi^\vee$  be the Hilbert newform associated to  $A^\vee$ , then  $(\phi^\vee, \phi^\vee)_{U_0(N)} = (\phi, \phi)_{U_0(N)}$ .

We may state the above theorem in simpler way under some assumptions. Assume that

- $\omega_A$  is unramified, and if  $v \in \Sigma_1$  then  $v \nmid c$ ;

Note that  $c_1 = c$  under the above assumption. Fix an infinite place  $\tau$  of  $F$  and let  $B$  be nearby quaternion algebra whose ramification set is obtained from the one of  $\mathbb{B}$  by removing  $\tau$ . Then there is an  $F$ -embedding of  $K$  into  $B$  which we fix once for all and view  $K^\times$  as a  $F$ -subtorus of  $B^\times$ . Let  $R$  be an admissible  $\mathcal{O}$ -order of  $B$  for  $(\pi, \chi)$ , by which we mean that  $\hat{R}$  is an admissible  $\hat{\mathcal{O}}$ -order of  $\mathbb{B}_f = \hat{B}$  for  $(\pi, \chi)$ . Note that  $R$  is of discriminant  $N$  such that  $R \cap K = \mathcal{O}_c$ . Let  $U = \hat{R}^\times \subset \hat{B}^\times$  and let  $X_U$  be the Shimura curve of level  $U$  so that it has complex uniformization

$$X_{U, \tau}(\mathbb{C}) = B_+^\times \backslash \mathcal{H} \times \hat{B}^\times / U \cup \{\text{Cusps}\},$$

where  $B_+^\times$  is the subgroup of elements  $x \in B^\times$  with totally positive norms. Let  $u = \#\kappa_c \cdot [\mathcal{O}_c^\times : \mathcal{O}^\times]$ . By Proposition 3.8, we have that  $V(\pi_A, \chi) \subset (\pi_A \otimes_M L)^{\hat{R}^\times}$ .

*Special case 1.* Further assume that  $(N, Dc) = 1$ . Then there is a non-constant morphism  $f : X_U \rightarrow A$  mapping a Hodge class on  $X_U$  to torsion of  $A$  and for any two such morphisms  $f_1, f_2 : X_U \rightarrow A$ ,  $n_1 f_1 = n_2 f_2$  for some non-zero integers  $n_1, n_2$ . Let  $P = [h_0, 1] \in X_U$  be the point with  $h_0$  the unique fixed point of  $K^\times$ . Replace  $\chi$  by  $\chi^{-1}$ , there is a non-constant morphism  $X_U \rightarrow A^\vee$  with similar uniqueness. For any such  $f_1 : X_U \rightarrow A$  and  $f_2 : X_U \rightarrow A^\vee$ , let  $(f_1, f_2) = f_1 \circ f_2^\vee$ . Then we have an equality in  $L \otimes_{\mathbb{Q}} \mathbb{C}$ :

$$L'(1, A, \chi) = \frac{(8\pi^2)^d (\phi, \phi)_{U_0(N)}}{u^2 \cdot \sqrt{|D_K| \|c^2\|}} \cdot \frac{\langle P_\chi^0(f_1), P_{\chi^{-1}}^0(f_2) \rangle_{K, L}}{(f_1, f_2)_U}.$$

*Special case 2.* Further assume that  $\omega_A$  is trivial, or more general, that  $\omega_A(\varpi_v) \in \text{Aut}(A)^2 \subset M^{\times 2}$  for all places  $v|(N, D)$  but  $v \nmid c$ , where  $\varpi_v$  is a uniformizer of  $F_v$ . For each place  $v|(N, D)$  but  $v \nmid c$ ,  $K_v^\times$  normalizes  $R_v^\times$  (See Lemma 3.4) and a uniformizer  $\varpi_{K_v}$  of  $K_v$  induces an automorphism  $T_{\varpi_{K_v}} : X_U \rightarrow X_U$  over  $F$ . Note that  $\chi_v(\varpi_{K_v}) \in \text{Aut}(A) \subset M^\times$ . There exists a non-constant morphism  $f : X_U \rightarrow A$  mapping a Hodge class to torsion point such that  $T_{\varpi_{K_v}} f = \chi^{-1}(\varpi_{K_v}) f$  for each place  $v|(N, D)$  but  $v \nmid c$ . Such  $f$  has the same uniqueness property as in special case 1. Then for any such  $f_1 : X_U \rightarrow A$  and  $f_2 : X_U \rightarrow A^\vee$ , we have an equality in  $L \otimes_{\mathbb{Q}} \mathbb{C}$ :

$$L'(\Sigma)(1, A, \chi) = 2^{-\#\Sigma_D} \cdot \frac{(8\pi^2)^d (\phi, \phi)_{U_0(N)}}{u^2 \cdot \sqrt{|D_K| \|c^2\|}} \cdot \frac{\langle P_\chi^0(f_1), P_{\chi^{-1}}^0(f_2) \rangle_{K, L}}{(f_1, f_2)_U},$$

where  $\Sigma$  is now the set of places  $v|(cD, N)$  of  $F$  such that if  $v|N$  then  $v \nmid D$ .

**Example.** Let  $\phi \in S_2(\Gamma_0(N))$  be a newform. Let  $K$  be an imaginary quadratic field of discriminant  $D$  and  $\chi$  a primitive character of  $\text{Pic}(\mathcal{O}_c)$ . Assume that  $(\phi, \chi)$  satisfies the Heegner condition (1)-(2) in Theorem 1.1, then by Lemma 3.1 (1) and (3),  $\epsilon(\phi, \chi) = -1$  and  $B = M_2(\mathbb{Q})$ . The condition (1)-(2) also implies that there exists  $a, b \in \mathbb{Z}$  with  $(N, a, b) = 1$  such that  $a^2 - 4Nb = Dc^2$ . Fix an embedding of  $K$  into  $B$  by

$$(Dc^2 + \sqrt{Dc^2})/2 \mapsto \begin{pmatrix} (Dc^2 + a)/2 & -1 \\ Nb & (Dc^2 - a)/2 \end{pmatrix}.$$

Then  $R := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \mid N|c \right\}$  is an order of  $B$  such that  $\hat{R} \cap K = \mathcal{O}_c$ . Let  $A$  be an abelian variety associated to  $\phi$  via Eichler-Shimura theory and  $f : X_0(N) \rightarrow A$  be any non-constant morphism mapping cusp  $\infty$  to  $O \in A$ . Then  $f \in V(\pi_A, \chi)$ . Let  $z \in \mathcal{H}$  be the fixed point by  $K^\times$ , then  $Nbz^2 - az + 1 = 0$ ,  $\mathcal{O}_c = \mathbb{Z} + \mathbb{Z}z^{-1}$ , and  $\mathfrak{n}^{-1} = \mathbb{Z} + \mathbb{Z}N^{-1}z^{-1}$  so that  $\mathcal{O}_c/\mathfrak{n} \cong \mathbb{Z}/N\mathbb{Z}$ . The point on  $X_0(N)$  corresponding  $z$  via complex uniformization represents the isogeny  $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}z) \rightarrow \mathbb{C}/(N^{-1}\mathbb{Z} + \mathbb{Z}z)$ , or  $\mathbb{C}/\mathcal{O}_c \rightarrow \mathbb{C}/\mathfrak{n}^{-1}$ . Thus Theorem 1.1 now follows from Theorem 1.5.

For various arithmetic applications, we may need explicit formula for different test vectors. We now give variations of the explicit formula for different test vectors. Let  $v$  be a finite place of  $F$ , fix  $\langle \cdot, \cdot \rangle_v$  a  $\mathbb{B}_v^\times$ -invariant pairing on  $\pi_{A,v} \times \pi_{A^\vee,v}$  and a Haar measure  $dt_v$  on  $F_v^\times \backslash K_v^\times$ . For any  $f'_{1,v} \in \pi_{A,v}, f'_{2,v} \in \pi_{A^\vee,v}$  with  $\langle f'_{1,v}, f'_{2,v} \rangle_v \neq 0$ , let

$$\beta^0(f'_{1,v}, f'_{2,v}) = \beta^0(f'_{1,v}, f'_{2,v}, dt_v) = \int_{F_v^\times \backslash K_v^\times} \frac{\langle \pi_{A,v}(t_v) f'_{1,v}, f'_{2,v} \rangle_v}{\langle f'_{1,v}, f'_{2,v} \rangle_v} \chi_v(t_v) dt_v.$$

For any two non-zero pure tensor forms  $f' = \otimes_v f'_v, f'' = \otimes_v f''_v \in \pi$ , we say that  $f', f''$  differ (resp. coincide) at a place  $v$  if  $f'_v$  and  $f''_v$  are not parallel (resp. are parallel). It is independent of the decompositions. In particular, if two non-zero pure tensor forms coincide locally everywhere then they are the same up to a scalar.

**Theorem 1.6** (Variation of Gross-Zagier Formula). *Let  $(A, \chi)$  and  $f_1 \in V(\pi_A, \chi), f_2 \in V(\pi_{A^\vee}, \chi^{-1})$  be as in Theorem 1.5. Let  $S$  be a finite set of finite places of  $F$ ,  $f'_1 \in \pi_A, f'_2 \in \pi_{A^\vee}$  be vectors such that  $f'_i$  and  $f_i$  coincide for any  $v \notin S, i = 1, 2$ , and  $\langle f'_{1,v}, f'_{2,v} \rangle_v \neq 0$  and  $\beta^0(f'_{1,v}, f'_{2,v}) \neq 0$ , for any  $v \in S$ . Define*

$$P_\chi^0(f'_1) = \frac{\#\text{Pic}(\mathcal{O}_{c_1})}{\text{Vol}(K^\times \widehat{F}^\times \backslash \widehat{K}^\times, dt)} \cdot \int_{K^\times \widehat{F}^\times \backslash \widehat{K}^\times} f'_1(P)^{\sigma_t} \chi(t) dt,$$

and define  $P_{\chi^{-1}}^0(f'_2)$  similarly. Then, with notations as in Theorem 1.5, we have that

$$L^{(\Sigma)}(1, A, \chi) = 2^{-\#\Sigma_D} \cdot \frac{(8\pi^2)^d \cdot (\phi, \phi)_{U_0(N)}}{u_1^2 \sqrt{|D_K| \|c_1^2\|}} \cdot \frac{\langle P_\chi^0(f'_1), P_{\chi^{-1}}^0(f'_2) \rangle_{K,L}}{(f'_1, f'_2)_{\mathcal{R}}^\times} \cdot \prod_{v \in S} \frac{\beta^0(f_{1,v}, f_{2,v})}{\beta^0(f'_{1,v}, f'_{2,v})},$$

which is independent of the choice of Haar measure  $dt_v$  for  $v \in S$ .

**Example.** Let  $A$  be the elliptic curve  $X_0(36)$  with the cusp  $\infty$  as the identity point and let  $K = \mathbb{Q}(\sqrt{-3})$ . Let  $p \equiv 2 \pmod{9}$  be a prime, then the field  $L' = K(\sqrt[3]{p})$  is contained in  $H_{3p}$ . Let  $\chi : \text{Gal}(L'/K) \rightarrow K^\times$  be the character mapping  $\sigma$  to  $(\sqrt[3]{p})^{\sigma-1}$ . Fix the embedding  $K \rightarrow M_2(\mathbb{Q})$  mapping  $w := (-1 + \sqrt{-3})/2$  to  $\begin{pmatrix} -1 & -p/6 \\ 6/p & 0 \end{pmatrix}$ .

For  $f' = \text{id} : X_0(36) \rightarrow A$ , let  $P \in X_0(36)$  be the point corresponding to  $-pw/6 \in \mathcal{H}$ . The Heegner divisor  $P_\chi^0(f')$  is

$$P_\chi^0(f') = \frac{1}{9} \sum_{t \in \text{Pic}(\mathcal{O}_{6p})} f'(P)^{\sigma_t} \chi(t).$$

One can show that  $P_\chi^0(f')$  is non-trivial (see [28], [12] and [5]) and then it follows that the prime  $p$  is the sum of two rational cubes. By the variation formula, one can easily obtain the height formula of  $P_\chi^0(f')$ : let  $\phi \in S_2(\Gamma_0(36))$  be the newform associated to  $A$ , and note that  $\#\Sigma_D = 1, u_1 = 1$ , and  $c_1 = p$  in the variation,

$$L^{(\infty)}(1, A, \chi) = 9 \cdot \frac{8\pi^2 \cdot (\phi, \phi)_{\Gamma_0(36)}}{\sqrt{3p^2}} \cdot \langle P_\chi^0(f'), P_{\chi^{-1}}^0(f') \rangle_{K,K}.$$

In fact,  $U = \mathcal{R}^\times$  in Theorem 1.5 is given by

$$\mathcal{R} = \left\{ \begin{pmatrix} a & b/6 \\ 6c & d \end{pmatrix} \in M_2(\widehat{\mathbb{Q}}) \mid a, b, c, d \in \widehat{\mathbb{Z}}, p^{-1}b + pc, a + pc - d \in 6\widehat{\mathbb{Z}} \right\}$$

and  $f \in V(\pi_A, \chi)$  is  $\chi_v^{-1}$ -eigen for  $v = 2, 3$ . Then  $(f', f') = \text{Vol}(X_U)/\text{Vol}(X_0(36)) = 2/9$ . The ratio  $\frac{\beta^0(f_v, f_v)}{\beta^0(f'_v, f'_v)}$  is equal to 1 at  $v = 2$  and 4 at  $v = 3$ .

**1.3. The Explicit Waldspurger Formula.** Let  $F$  be a general base number field. Let  $B$  be a quaternion algebra over  $F$  and  $\pi$  a cuspidal automorphic representation of  $B_\mathbb{A}^\times$  with central character  $\omega$ . Let  $K$  be a quadratic field extension of  $F$  and  $\eta$  the quadratic Hecke character on  $F^\times \backslash \mathbb{A}^\times$  associated to the quadratic extension. Let  $\chi$  be a Hecke character on  $K_\mathbb{A}^\times$ . Write  $L(s, \pi, \chi)$  for the Rankin L-series  $L(s, \pi^{\text{JL}} \times \pi_\chi)$ , where  $\pi^{\text{JL}}$  is the Jacquet-Langlands correspondence of  $\pi$  on  $\text{GL}_2(\mathbb{A})$  and  $\pi_\chi$  the automorphic representation of  $\text{GL}_2(\mathbb{A})$  corresponding to theta series of  $\chi$  so that  $L(s, \pi_\chi) = L(s, \chi)$ . Assume that

$$\omega \cdot \chi|_{\mathbb{A}^\times} = 1.$$



Then for any place  $v$  of  $F$ , the local root number  $\epsilon(1/2, \pi_v, \chi_v)$  of the Rankin L-series is independent of the choice of additive character. We also assume that for all places  $v$  of  $F$

$$\epsilon(1/2, \pi_v, \chi_v) = \chi_v \eta_v(-1) \epsilon(B_v),$$

where  $\epsilon(B_v) = -1$  if  $B_v$  is division and  $+1$  otherwise. It follows that the global root number  $\epsilon(1/2, \pi, \chi) = +1$  and there exists an  $F$ -embedding of  $K$  into  $B$ . We fix such an embedding once for all and view  $K^\times$  as an  $F$ -subtorus of  $B^\times$ .

Let  $N$  be the conductor of  $\pi^{\text{JL}}$ ,  $D$  the relative discriminant of  $K$  over  $F$ ,  $c \subset \mathcal{O}$  be the ideal maximal such that  $\chi$  is trivial on  $\prod_{v \nmid c} \mathcal{O}_{K_v}^\times \prod_{v|c} (1 + c\mathcal{O}_{K,v})$ . Define the following sets of places  $v$  of  $F$  dividing  $N$ :

$$\Sigma_1 := \{v|N \text{ nonsplit in } K : \text{ord}_v(c) < \text{ord}_v(N)\},$$

Let  $c_1 = \prod_{\mathfrak{p}|c, \mathfrak{p} \notin \Sigma_1} \mathfrak{p}^{\text{ord}_{\mathfrak{p}} c}$  be the  $\Sigma_1$ -off part of  $c$ ,  $N_1$  the  $\Sigma_1$ -off part of  $N$ , and  $N_2 = N/N_1$  be the  $\Sigma_1$ -part of  $N$ .

Let  $R$  be an admissible  $\mathcal{O}$ -order of  $B$  for  $(\pi, \chi)$  in the sense that  $R_v$  is admissible for  $(\pi_v, \chi_v)$  for every finite place  $v$  of  $F$ . It follows that  $R$  is  $\mathcal{O}$ -order with discriminant  $N$  such that  $R \cap K = \mathcal{O}_{c_1}$ .

Let  $U = \prod_v U_v \subset B_{\mathbb{A}}^\times$  be a compact subgroup satisfying that for any finite place  $v$ ,  $U_v = R_v^\times$ , and that for any infinite place  $v$  of  $F$ ,  $U_v$  is a maximal compact subgroup of  $B_v^\times$  such that  $U_v \cap K_v^\times$  is the maximal compact subgroup of  $K_v^\times$ . Note that for any finite place  $v|N_1$ ,  $B_v$  must be split. Let  $Z \cong \mathbb{A}_f^\times$  denote the center of  $\widehat{B}^\times$ . The group  $U^{(N_2\infty)}$  has a decomposition  $U^{(N_2\infty)} = U' \cdot (Z \cap U^{(N_2\infty)})$  where  $U' = \prod_{v \nmid N_2\infty} U'_v$  such that for any finite place  $v \nmid N_2$ ,  $U'_v = U_v$  if  $v \nmid N$  and  $U'_v \cong U_1(N)_v$  otherwise. View  $\omega$  as a character on  $Z$  and we may define a character on  $U^{(c_2\infty)}$  by  $\omega$  on  $Z \cap U^{(c_2\infty)}$  and trivial on  $U'$ , which we also denote by  $\omega$ .

**Definition 1.7.** Let  $V(\pi, \chi)$  denote the space of forms  $f = \otimes_v f_v \in \pi$  such that  $f$  is  $\omega$ -eigen under  $U^{(N_2\infty)}$ ; for all places  $v \in \Sigma_1$ ,  $f$  is  $\chi_v^{-1}$ -eigen under  $K_v^\times$ ; and for any infinite place  $v$ ,  $f$  is  $\chi_v^{-1}$ -eigen under  $U_v \cap K_v^\times$  with weight minimal. The space  $V(\pi, \chi)$  is actually a one dimensional space (see Proposition 3.7).

Let  $r, s, t$  be integers such that  $B \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{H}^r \times M_2(\mathbb{R})^s \times M_2(\mathbb{C})^t$ , and let  $X_U$  denote the  $U$ -level real manifold

$$X_U = B_+^\times \backslash (\mathcal{H}_2^s \times \mathcal{H}_3^t) \times \widehat{B}^\times / U,$$

which has finitely many connected components, where  $\mathcal{H}_2, \mathcal{H}_3$  are the usual hyperbolic spaces of dimension two and three respectively. Define the volume of  $X_U$ , denoted by  $\text{Vol}(X_U)$ , as follows.

- if  $s + t > 0$ , then  $X_U$  is disjoint union of dimension  $2s + 3t$  manifolds:

$$X_U = B_+^\times \backslash (\mathcal{H}_2^s \times \mathcal{H}_3^t) \times \widehat{B}^\times / U = \bigsqcup_i \Gamma_i \backslash (\mathcal{H}_2^s \times \mathcal{H}_3^t),$$

for some discrete subgroup  $\Gamma_i \subset B_+^\times \cap \prod_{v|\infty, B_v \text{ not division}} (B_v)^\times$ , then define volume of  $X_U$  with the measure  $dx dy / (4\pi y^2)$  on  $\mathcal{H}_2$  and the measure  $dx dy dv / \pi^2 v^3$  on  $\mathcal{H}_3$ . Here the notation  $\mathcal{H}_3$  is the same as in [37].

- if  $s + t = 0$ , then  $F$  is totally real and  $B$  is totally definite. For any open compact subgroup  $U$  of  $\widehat{B}^\times$ , the double coset  $B^\times \backslash \widehat{B}^\times / U$  is finite, let  $g_1, \dots, g_n \in \widehat{B}^\times$  be a complete set of representatives for the coset. Let  $\mu_Z = \widehat{F}^\times \cap U$ , then for any  $g \in \widehat{B}^\times$ ,  $B^\times \cap gUg^{-1} / \mu_Z$  is a finite set. Define the volume of  $X_U$  to be the Mass of  $U$

$$\text{Vol}(X_U) = \text{Mass}(U) = \sum_{i=1}^n \frac{1}{\#(B^\times \cap g_i U g_i^{-1} / \mu_Z)}.$$

For any automorphic forms  $f_1 \in \pi$  and  $f_2 \in \tilde{\pi}$ ,  $\langle f_1, f_2 \rangle_{\text{Pet}}$  is the Petersson pairing of  $f_1, f_2$  defined by

$$\langle f_1, f_2 \rangle_{\text{Pet}} = \int_{B^\times \mathbb{A}^\times \backslash B_{\mathbb{A}}^\times} f_1(g) f_2(g) dg,$$

where  $dg$  is the Tamagawa measure on  $F^\times \backslash B^\times$  so that  $B^\times \mathbb{A}^\times \backslash B_{\mathbb{A}}^\times$  has total volume 2. For any  $f_1 \in V(\pi, \chi)$  and  $f_2 \in V(\tilde{\pi}, \chi^{-1})$ , one may define the  $U$ -level pairing as

$$\langle f_1, f_2 \rangle_U = \frac{\langle f_1, f_2 \rangle_{\text{Pet}}}{2} \cdot \text{Vol}(X_U).$$

For any  $f \in V(\pi, \chi)$ , define the  $c_1$ -level period of  $f \in V(\pi, \chi)$  as follows. Let  $\overline{K_\infty^\times/F_\infty^\times}$  be the closure of  $K_\infty^\times/F_\infty^\times$  in the compact group  $K_\mathbb{A}^\times/\mathbb{A}^\times K^\times$  and endowed on  $\overline{K_\infty^\times/F_\infty^\times}$  the Haar measure  $dh$  of total volume one, let

$$P_\chi^0(f) = \sum_{t \in \text{Pic}_{K/F}(\mathcal{O}_{c_1})} f^0(t)\chi(t), \quad f^0(t) = \int_{\overline{K_\infty^\times/F_\infty^\times}} f(th)\chi(h)dh.$$

Note that the function  $f^0(t)\chi(t)$  on  $K_\mathbb{A}^\times$  is constant on  $K_{\Sigma_1}^\times$  and then can be viewed as a function on  $\text{Pic}_{K/F}(\mathcal{O}_{c_1}) = \widehat{K}^\times/K^\times \widehat{F}^\times \widehat{\mathcal{O}_{c_1}^\times}$ . Note that when  $F$  is totally real and all infinite places  $v$  of  $F$  are inert in  $K$ ,  $f^0 = f$ .

**Notations:** Let  $b$  be an integral ideal of  $F$ , we define the relative regulator  $R_b$  to be the quotient of the regulator of  $\mathcal{O}_b^\times$  by the regulator of  $\mathcal{O}^\times$  and  $w_b = \#\mathcal{O}_{b,\text{tor}}^\times/\#\mathcal{O}_{\text{tor}}^\times$ . Denote by  $\kappa_b$  the kernel of the natural homomorphism from  $\text{Pic}(\mathcal{O})$  to  $\text{Pic}(\mathcal{O}_b)$ . Define  $\nu_b = 2^{-r_{K/F}} R_b^{-1} \cdot \#\kappa_b \cdot w_b$  where  $r_{K/F} = \text{rank}\mathcal{O}_K^\times - \text{rank}\mathcal{O}^\times$ . For example, if  $F$  is a totally real field of degree  $d$  and  $K$  is a totally imaginary quadratic field extension over  $F$ , then  $\nu_b = 2^{1-d} \cdot \#\kappa_b \cdot [\mathcal{O}_b^\times : \mathcal{O}^\times]$ , where  $\kappa_b \subset \kappa_1$  and  $\#\kappa_1 = 1$  or  $2$  by Theorem 10.3 of [39].

For an infinite place  $v$  of  $F$ , let  $U_v$  denote the maximal compact subgroup of  $\text{GL}_2(F_v)$ , which is  $O_2$  if  $v$  is real and  $U_2$  if  $v$  is complex, and let  $U_{1,v} \subset U_v$  denote its subgroup of diagonal matrices  $\begin{pmatrix} a & \\ & 1 \end{pmatrix}$  for  $a \in F_v^\times$  with  $|a|_v = 1$ . For a generic  $(\mathfrak{g}_v, U_v)$ -module  $\sigma_v$  and a non-trivial additive character  $\psi_v$  of  $F_v$ , let  $\mathcal{W}(\sigma_v, \psi_v)$  be the  $\psi_v$ -Whittaker model of  $\sigma_v$ . There is an invariant bilinear pairing on  $\mathcal{W}(\sigma_v, \psi_v) \times \mathcal{W}(\widetilde{\sigma}_v, \psi_v^{-1})$ ,

$$\langle W_1, W_2 \rangle_v := \int_{F_v^\times} W_1 \left[ \begin{pmatrix} a & \\ & 1 \end{pmatrix} \right] W_2 \left[ \begin{pmatrix} a & \\ & 1 \end{pmatrix} \right] d^\times a$$

with the measure  $d^\times a = L(1, 1_v) \frac{da}{|a|_v}$  where  $da$  equals  $[F_v : \mathbb{R}]$  times the usual Lebesgue measure on  $F_v$ . Let  $W_0 \in \mathcal{W}(\sigma_v, \psi_v)$  be the vector invariant under  $U_{1,v}$  with minimal weight such that

$$L(s, \pi_v) = Z(s, W_0), \quad Z(s, W_0) := \int_{F_v^\times} W_{\sigma_v} \left[ \begin{pmatrix} a & \\ & 1 \end{pmatrix} \right] |a|_v^{s-1/2} d^\times a$$

where  $d^\times a$  is Tamagawa measure. Similarly define  $\widetilde{W}_0$  for  $\widetilde{\sigma}_v$ . Then  $\Omega_{\sigma_v} := \langle W_0, \widetilde{W}_0 \rangle_v$  is an invariant of  $\sigma_v$  which is independent of the choice of  $\psi_v$  (see an explicit formula for  $\Omega_{\sigma_v}$  before Lemma 3.14). We associate  $(\sigma_v, \chi_v)$  a constant by

$$(1.1) \quad C(\sigma_v, \chi_v) := \begin{cases} 2^{-1} \pi \cdot \Omega_{\sigma_v}^{-1}, & \text{if } K_v \text{ nonsplit;} \\ \Omega_{\sigma_v \otimes \chi_{1,v}} \cdot \Omega_{\sigma_v}^{-1}, & \text{if } K_v \text{ split,} \end{cases}$$

where for split  $K_v \cong F_v^2$ , embedded into  $M_2(F_v)$  diagonally, the character  $\chi_1$  is given by  $\chi_{1,v}(a) := \chi_v \left[ \begin{pmatrix} a & \\ & 1 \end{pmatrix} \right]$ . If  $v$  is a real place of  $F$  and  $\sigma_v$  is a discrete series of weight  $k$ , then  $C(\sigma_v, \chi_v) = 4^{k-1} \pi^{k+1} \Gamma(k)^{-1}$  when  $K_v \cong \mathbb{C}$ , and  $C(\sigma_v, \chi_v) = 1$  when  $K_v \cong \mathbb{R}^2$ .

Let  $\sigma$  be the Jacquet-Langlands correspondence of  $\pi$  to  $\text{GL}_2(\mathbb{A})$ , the normalized new vector  $\phi^0 = \otimes_v \phi_v \in \sigma$  is the one fixed by  $U_1(N)$  and  $\phi_v$  is fixed by  $U_{1,v}$  with weight minimal for all  $v|\infty$  such that

$$L(s, \sigma) = |\delta|_\mathbb{A}^{s-\frac{1}{2}} Z(s, \phi^0), \quad Z(s, \phi^0) := \int_{F^\times \backslash \mathbb{A}^\times} \phi^0 \left( \begin{pmatrix} a & \\ & 1 \end{pmatrix} \right) |a|_\mathbb{A}^{s-\frac{1}{2}} d^\times a,$$

with Tamagawa measure on  $\mathbb{A}^\times$  so that  $\text{Res}_{s=1} \int_{|a| \leq 1, a \in F^\times \backslash \mathbb{A}^\times} |a|^{s-1} d^\times a = \text{Res}_{s=1} L(s, 1_F)$ . Note that when  $F$  is a totally real field and  $\sigma$  a cuspidal automorphic representation such that  $\sigma_v$  is discrete series for any infinite place  $v$ , the normalized new vector  $\phi^0$  is not parallel to the Hilbert newform  $\phi$ : they are different at infinity. Note that if  $\sigma$  is unitary and  $\phi^0$  is the normalized new vector of  $\sigma$ , then  $\bar{\sigma} \cong \widetilde{\sigma}$  and  $\bar{\phi}^0$  is the normalized new vector of  $\bar{\sigma}$ . We will see that  $(\phi, \phi)_{U_0(N)} = (2\pi)^d \langle \phi_0, \bar{\phi}_0 \rangle_{U_0(N)}$ .

**Theorem 1.8** (Explicit Waldspurger Formula). *Let  $F$  be a number field. Let  $B$  be a quaternion algebra over  $F$  and  $\pi$  an irreducible cuspidal automorphic representation of  $B_\mathbb{A}^\times$  with central character  $\omega$ . Let  $K$  be a quadratic field extension of  $F$  and  $\chi$  a Hecke character of  $K_\mathbb{A}^\times$ . Assume that*

- (1)  $\omega \cdot \chi|_{\mathbb{A}^\times} = 1$ ,
- (2)  $\epsilon(1/2, \pi_v, \chi_v) = \chi_v \eta_v(-1) \epsilon(B_v)$  for all places  $v$  of  $F$ .

Then for any non zero forms  $f_1 \in V(\pi, \chi)$  and  $f_2 \in V(\tilde{\pi}, \chi^{-1})$ , we have

$$L^{(\Sigma)}(1/2, \pi, \chi) = 2^{-\#\Sigma_D+2} \cdot C_\infty \cdot \frac{\langle \phi_1^0, \phi_2^0 \rangle_{U_0(N)}}{\nu_{c_1}^2 \sqrt{|D_K| \|c_1\|^2}} \cdot \frac{P_\chi^0(f_1) P_{\chi^{-1}}^0(f_2)}{\langle f_1, f_2 \rangle_{\hat{R}^\times}},$$

where  $\phi_1^0 \in \pi^{\text{JL}}$  and  $\phi_2^0 \in \tilde{\pi}^{\text{JL}}$  are normalized new vectors,  $\Sigma$  is the set of places  $v|(cD, N)_\infty$  of  $F$  such that if  $v|N$  then  $\text{ord}_v(c/N) \geq 0$  and if  $v|\infty$  then  $K_v \cong \mathbb{C}$ . The constant  $C_\infty = \prod_{v|\infty} C_v$ ,  $c_1|c$  and  $\Sigma_D$  are the same as in Theorem 1.5, and  $C_v = C(\pi_v^{\text{JL}}, \chi_v)$  is given in (1.1).

For many applications, we need explicit form of Waldspurger formula for different test vectors. The following variation formula is useful. For each place  $v$  of  $F$ , fix a  $B_v^\times$ -invariant pairing  $\langle \cdot, \cdot \rangle_v$  on  $\pi_v \times \tilde{\pi}_v$ . Here if  $v|\infty$ , we mean it is the restriction of a  $B_v^\times$ -invariant pairing on the corresponding smooth representations. For any  $f'_{1,v} \in \pi_v, f'_{2,v} \in \tilde{\pi}_v$  with  $\langle f'_{1,v}, f'_{2,v} \rangle_v \neq 0$ , define  $\beta^0(f'_{1,v}, f'_{2,v})$  same as in Theorem 1.6.

**Theorem 1.9** (Variation of Waldspurger Formula). *Let  $(\pi, \chi)$  and  $f_1 \in V(\pi, \chi), f_2 \in V(\tilde{\pi}, \chi^{-1})$  be as in Theorem 1.8. Let  $S$  be a finite set of places of  $F$ ,  $f'_1 \in \pi, f'_2 \in \tilde{\pi}$  be pure vectors which coincide with  $f_1, f_2$  outside  $S$  respectively, such that  $\langle f'_{1,v}, f'_{2,v} \rangle_v \neq 0$  and  $\beta^0(f'_{1,v}, f'_{2,v}) \neq 0$  for all  $v \in S$ . Here  $\beta^0$  is similarly defined as in Theorem 1.6. Define*

$$P_\chi^0(f'_1) = \frac{\#\text{Pic}_{K/F}(\mathcal{O}_{c_1})}{\text{Vol}(K^\times \mathbb{A}^\times \backslash K_\mathbb{A}^\times, dt)} \cdot \int_{K^\times \mathbb{A}^\times \backslash K_\mathbb{A}^\times} f'_1(t) \chi(t) dt,$$

and define  $P_{\chi^{-1}}^0(f'_2)$  similarly. Then we have that

$$L^{(\Sigma)}(1/2, \pi, \chi) = 2^{-\#\Sigma_D+2} \cdot C_\infty \cdot \frac{\langle \phi_1^0, \phi_2^0 \rangle_{U_0(N)}}{\nu_{c_1}^2 \sqrt{|D_K| \|c_1\|^2}} \cdot \frac{P_\chi^0(f'_1) P_{\chi^{-1}}^0(f'_2)}{\langle f'_1, f'_2 \rangle_{\hat{R}^\times}} \cdot \prod_{v \in S} \frac{\beta^0(f_{1,v}, f_{2,v})}{\beta^0(f'_{1,v}, f'_{2,v})},$$

where the notations are the same as in Theorem 1.8.

**Example.** Let  $\phi = \sum a_n q^n \in S_2(\Gamma_0(N))$  be a newform of weight 2 and  $p$  a good ordinary prime of  $\phi$ ,  $K$  an imaginary quadratic field of discriminant  $D$  and  $\chi$  a character of  $\text{Gal}(H_c/K)$  of conductor  $c$  prime to  $p$ . Assume that the conditions (i)-(ii) in Theorem 1.2 are satisfied. Let  $B$  be the quaternion algebra,  $\pi$  the cuspidal automorphic representation on  $B_\mathbb{A}^\times$  and identify  $\tilde{\pi}$  with  $\bar{\pi}$ , and  $f \in \pi^{\hat{R}^\times} = V(\pi, \chi)$  a non-zero test vector as in Theorem 1.8. Define the  $p$ -stabilization of  $f$  by

$$f^\dagger = f - \alpha^{-1} \pi \begin{pmatrix} 1 & \\ & p \end{pmatrix} f$$

where  $\alpha$  is the unit root of  $X^2 - a_p X + p$  and let  $\beta = p/\alpha$  is another root. By the above variation formula and Theorem 1.2, one may easily obtain formula for  $P_\chi^0(f^\dagger)$  which is used to give interpretation property of anticyclotomic  $p$ -adic L-function.

$$L(1, \phi, \chi) = 2^{-\mu(N, D)} \cdot \frac{8\pi^2(\phi, \phi)_{\Gamma_0(N)}}{[\mathcal{O}_c^\times : \mathbb{Z}^\times]^2 \sqrt{|Dc^2|}} \cdot \frac{|P_\chi^0(f^\dagger)|^2}{\langle f^\dagger, f^\dagger \rangle_{\hat{R}^\times}} \cdot e_p,$$

where

$$e_p = \frac{\beta^0(W, \overline{W})}{\beta^0(W^\dagger, \overline{W^\dagger})} = \frac{L(2, 1_p)}{L(1, \pi_p, \text{ad})} \cdot (1 - \alpha^{-1} \chi_1(p))^{-1} (1 - \beta^{-1} \chi_1^{-1}(p))^{-1}.$$

Here  $W$  is a new vector of the Whittaker model  $\mathcal{W}(\pi_p, \psi_p)$  with  $\psi_p(x) = e^{-2\pi i \iota(x)}$  where  $\iota : \mathbb{Q}_p/\mathbb{Z}_p \rightarrow \mathbb{Q}/\mathbb{Z}$  is the natural embedding, and  $W^\dagger := W - \alpha^{-1} \pi_p \begin{pmatrix} 1 & \\ & p \end{pmatrix} W$  is its stabilization, where  $K_p^\times \cong \mathbb{Q}_p^{\times 2}$  is embedded into  $\text{GL}_2(\mathbb{Q}_p)$  as diagonal subgroup and  $\chi_1(a) = \chi \begin{pmatrix} a & \\ & 1 \end{pmatrix}$ .

Now we consider the situation that

- 1)  $F$  is a totally real field and  $K$  is a totally imaginary quadratic extension over  $F$ ,
- 2) for any place  $v|\infty$  of  $F$ ,  $\pi_v^{\text{JL}}$  is a unitary discrete series of weight 2,
- 3)  $(c, N) = 1$ .

Now let  $\phi$  be the Hilbert newform as in Theorem 1.5, (which is different from the one we choose in Theorem 1.8). We are going to give an explicit form of Waldspurger formula following Gross [14], which is quoted in many references. Let  $X = B^\times \backslash \hat{B}^\times / \hat{R}^\times$  and let  $g_1, \dots, g_n \in \hat{B}^\times$  be a complete set of representatives of  $X$ . Denote  $[g] \in X$  for the class of an element  $g \in \hat{B}^\times$ . Note that for each  $g_i$ , let

$\Gamma_i = (B^\times \cap g_i \widehat{R}^\times g_i^{-1}) / \mathcal{O}^\times$ , which is finite and denote by  $w_i$  its order. Let  $\mathbb{Z}[X]$  be the free  $\mathbb{Z}$ -module (of rank  $\#X$ ) of formal sums  $\sum_i a_i [g_i]$ . There is a height pairing on  $\mathbb{Z}[X] \times \mathbb{Z}[X]$  defined by

$$\langle \sum a_i [g_i], \sum b_i [g_i] \rangle = \sum_i a_i b_i w_i.$$

By Eichler's norm theorem, the norm map

$$N : X := B^\times \backslash \widehat{B}^\times / \widehat{R}^\times \longrightarrow C_+ := F_+^\times \backslash \widehat{F}^\times / \widehat{\mathcal{O}}^\times$$

is surjective. For each  $c \in C_+$ , let  $X_c \subset X$  be the preimage of  $c$  and  $\mathbb{Z}[X_c]$  be the submodule of  $\mathbb{Z}[X]$  supported on  $X_c$ . Then  $\mathbb{Z}[X] = \bigoplus_{c \in C_+} \mathbb{Z}[X_c]$ . Let  $\mathbb{Z}[X_c]^0$  be the submodule of classes  $\sum a_i [g_i] \in \mathbb{Z}[X_c]$  with degree  $\sum_i a_i = 0$ , and let  $\mathbb{Z}[X]^0 = \bigoplus_{c \in C_+} \mathbb{Z}[X_c]^0$  and  $\mathbb{C}[X]^0 = \mathbb{Z}[X]^0 \otimes_{\mathbb{Z}} \mathbb{C}$ . Note that  $V(\pi, \chi) \subset \pi^{\widehat{R}^\times}$  by Proposition 3.8, and then there is an injection

$$V(\pi, \chi) \longrightarrow \mathbb{C}[X]^0, \quad f \mapsto \sum f([g_i]) w_i^{-1} [g_i],$$

and view  $V(\pi, \chi)$  be a line on  $\mathbb{C}[X]^0$ . It follows that  $\langle f, f \rangle = \langle f, f \rangle_{\widehat{R}^\times}$ . The fixed embedding  $K \rightarrow B$  induces a map

$$\text{Pic}(\mathcal{O}_c) \longrightarrow X, \quad t \mapsto x_t,$$

using which we define an element in  $\mathbb{C}[X]$ ,

$$P_\chi := \sum_{t \in \text{Pic}(\mathcal{O}_c)} \chi^{-1}(t) x_t$$

and let  $P_\chi^\pi$  be its projection to the line  $V(\pi, \chi)$ . Then the Explicit formula in Theorem 1.8 implies

**Theorem 1.10.** *Let  $(\pi, \chi)$  be as above with conditions 1)-3). The height of  $P_\chi^\pi$  is given by the formula*

$$L^{(\Sigma)}(1/2, \pi, \chi) = 2^{-\#\Sigma_D} \cdot \frac{(8\pi^2)^d \cdot (\phi, \phi)_{U_0(N)}}{u^2 \sqrt{|D_K|} \|c\|^2} \cdot \langle P_\chi^\pi, P_\chi^\pi \rangle,$$

where

$$\Sigma := \{v | (N, D) \infty, \text{ if } v | N \text{ then } v \nmid D\}, \quad \Sigma_D := \{v | (N, D)\},$$

$u = \#\kappa_c \cdot [\mathcal{O}_c^\times : \mathcal{O}^\times]$ , and  $\phi \in \pi^{\text{JL}}$  is the Hilbert newform as in Theorem 1.5. For any non-zero vector  $f \in V(\pi, \chi)$ , let  $P_\chi^0(f) = \sum_{t \in \text{Pic}(\mathcal{O}_c)} f(t) \chi(t)$ , then we have

$$L^{(\Sigma)}(1/2, \pi, \chi) = 2^{-\#\Sigma_D} \cdot \frac{(8\pi^2)^d \cdot (\phi, \phi)_{U_0(N)}}{u^2 \sqrt{|D_K|} \|c\|^2} \cdot \frac{|P_\chi^0(f)|^2}{\langle f, f \rangle}.$$

**Remark.** When  $c$  and  $N$  have common factor, one can still formulate an explicit formula in the spirit of Gross by defining a system of height pairings  $\langle \cdot, \cdot \rangle_U$  in the same way of Theorem 1.8.

As a byproduct, we obtain the following theorem about the relation between Petersson norm of newform and a special value of adjoint L-function.

**Proposition 1.11.** *Let  $F$  be a totally real field and  $\sigma$  a cuspidal unitary automorphic representation of  $\text{GL}_2(\mathbb{A})$  of conductor  $N$  such that for any  $v | \infty$ ,  $\sigma_v$  is a discrete series of weight  $k_v$ . Let  $\phi$  be the Hilbert newform in  $\sigma$  as in Theorem 1.5. Then we have that*

$$\frac{L^{(S)}(1, \sigma, \text{ad})}{(\phi, \phi)_{U_0(N)}} = 2^{d-1+\sum_{v|\infty} k_v} \cdot \|N\delta^{-2}\|^{-1} \cdot h_F^{-1},$$

where  $S$  is the set of finite places  $v$  of  $F$  with  $\text{ord}_v(N) \geq 2$ ,  $h_F$  is the ideal class number of  $F$ , and

$$(\phi, \phi)_{U_0(N)} = \iint_{X_{U_0(N)}} |\phi|^2 \left( \bigwedge_{v|\infty} y_v^{k_v-2} dx_v dy_v \right), \quad z_v = x_v + y_v i.$$

Or equivalently,

$$\frac{L^{(N\infty)}(1, \sigma, \text{ad})}{(\phi, \phi)_{U_0(N)}} = \frac{1}{2} \cdot \|N\delta^{-2}\|^{-1} \cdot h_F^{-1} \cdot \prod_{v|\infty} \frac{4^{k_v} \pi^{k_v+1}}{\Gamma(k_v)} \cdot \prod_{v \nmid N} L(2, 1_{F_v})^{-1}.$$

*Proof.* This follows from Proposition 2.1, Lemma 2.2, and Proposition 3.11. □

**Example.** Assume that  $F = \mathbb{Q}$  and  $\sigma$  is the cuspidal automorphic representation associated to a cuspidal newform  $\phi \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ . Then we have that

$$L(1, \sigma, \mathrm{ad}) = 2^k \cdot (\phi, \phi)_{\mathrm{SL}_2(\mathbb{Z})}, \quad L^{(\infty)}(1, \sigma, \mathrm{ad}) = \frac{2^{2k-1} \pi^{k+1}}{\Gamma(k)} \cdot (\phi, \phi)_{\mathrm{SL}_2(\mathbb{Z})}.$$

## 2. REDUCTION TO LOCAL THEORY

We now explain how to obtain these explicit formulas in Theorems 1.5 and 1.8 from the original Waldspurger formula and the general Gross-Zagier formula proved by Yuan-Zhang-Zhang in [40]. We first consider Waldspurger formula. Let  $B$  be a quaternion algebra over a number field  $F$  and  $\pi$  a cuspidal automorphic representation on  $B_{\mathbb{A}}^{\times}$  with central character  $\omega$ . Let  $K$  be a quadratic field extension over  $F$  and  $\chi$  be a Hecke character on  $K_{\mathbb{A}}^{\times}$ . Assume that (1)  $\omega \cdot \chi|_{\mathbb{A}^{\times}} = 1$ , (2) for any place  $v$  of  $F$ ,  $\epsilon(1/2, \pi_v, \chi_v) = \chi_v \eta_v(-1) \epsilon(B_v)$ . Define Petersson pairing on  $\pi \otimes \tilde{\pi}$  by

$$\langle f_1, f_2 \rangle_{\mathrm{Pet}} = \int_{B^{\times} \mathbb{A}^{\times} \backslash B_{\mathbb{A}}^{\times}} f_1(g) f_2(g) dg$$

with Tamagawa measure so that the volume of  $B^{\times} \mathbb{A}^{\times} \backslash B_{\mathbb{A}}^{\times}$  is 2. Let  $P_{\chi}$  denote the period functional on  $\pi$ :

$$P_{\chi}(f) = \int_{K^{\times} \mathbb{A}^{\times} \backslash K_{\mathbb{A}}^{\times}} f(t) \chi(t) dt, \quad \forall f \in \pi.$$

Then Waldspurger's period formula ([38] or Theorem 1.4 of [40]) says that for any pure tensors  $f_1 \in \pi, f_2 \in \tilde{\pi}$  with  $\langle f_1, f_2 \rangle_{\mathrm{Pet}} \neq 0$ ,

$$(2.1) \quad \frac{P_{\chi}(f_1) P_{\chi^{-1}}(f_2)}{\langle f_1, f_2 \rangle_{\mathrm{Pet}}} = \frac{L(1/2, \pi, \chi)}{2L(1, \pi, \mathrm{ad}) L(2, 1_F)^{-1}} \cdot \prod_v \beta_v(f_{1,v}, f_{2,v}),$$

where  $L(1, \pi, \mathrm{ad})$  is defined using the Jacquet-Langlands lifting of  $\pi$ , and for any place  $v$  of  $F$ , let  $\langle \cdot, \cdot \rangle_v : \pi_v \times \tilde{\pi}_v \rightarrow \mathbb{C}$  be a non-trivial invariant pairing, and

$$\beta(f_{1,v}, f_{2,v}) = \frac{L(1, \eta_v) L(1, \pi_v, \mathrm{ad})}{L(1/2, \pi_v, \chi_v) L(2, 1_{F_v})} \int_{K_v^{\times} / F_v^{\times}} \frac{\langle \pi(t_v) f_{1,v}, f_{2,v} \rangle_v}{\langle f_{1,v}, f_{2,v} \rangle_v} \chi(t_v) dt_v,$$

Here local Haar measures  $dt_v$  are chosen such that  $\otimes_v dt_v = dt$  is the Haar measure on  $K_{\mathbb{A}}^{\times} / \mathbb{A}^{\times}$  in the definitions of  $P_{\chi}$  and  $P_{\chi^{-1}}$ , and the volume of  $K^{\times} \backslash K_{\mathbb{A}}^{\times} / \mathbb{A}^{\times}$  with respect to  $dt$  is  $2L(1, \eta)$ . Note that the Haar measure  $dt$  is different from the one used in Theorem 1.4 of [40]. To obtain the explicit formula, we first relate  $P_{\chi}(f)$ ,  $L(1, \pi, \mathrm{ad})$ , and  $\langle f_1, f_2 \rangle_{\mathrm{Pet}}$  to the corresponding objects with levels in Theorem 1.8, and reduce to local computation.

For our purpose, it is more convenient to normalize local additive characters and local Haar measures as follows. Take the additive character  $\psi = \otimes_v \psi_v$  on  $\mathbb{A}$  as follows:

$$\psi_v(a) = \begin{cases} e^{2\pi i a}, & \text{if } F_v = \mathbb{R}; \\ e^{4\pi i \mathrm{Re}(a)}, & \text{if } F_v = \mathbb{C}; \\ \psi_p(\mathrm{tr}_{F/\mathbb{Q}_p}(a)), & \text{if } F_v \text{ is a finite extension over } \mathbb{Q}_p \text{ for some prime } p, \end{cases}$$

where  $\psi_p(b) = e^{-2\pi i \iota(b)}$  and  $\iota : \mathbb{Q}_p / \mathbb{Z}_p \rightarrow \mathbb{Q} / \mathbb{Z}$  is the natural embedding. It turns out that  $\psi$  is a character on  $F \backslash \mathbb{A}$ . For any place  $v$  of  $F$ , let  $da_v$  denote the Haar measure on  $F_v$  self-dual to  $\psi_v$ , and let  $d^{\times} a_v$  denote the Haar measure on  $F_v^{\times}$  defined by  $d^{\times} a_v = L(1, 1_v) \frac{da_v}{|a_v|_v}$ . Let  $L$  be a separable quadratic extension of  $F_v$  or a quaternion algebra over  $F_v$  and  $q$  the reduced norm on  $L$ , then  $(L, q)$  is a quadratic space over  $F_v$ . Fix the Haar measure  $dx$  on  $L$  to be the one self-dual with respect to  $\psi_v$  and  $q$  in the sense that  $\widehat{\Phi}(x) = \Phi(-x)$  for any  $\Phi \in S(L)$ , where  $\widehat{\Phi}(y) := \int_L \Phi(x) \psi_v(\langle x, y \rangle) dx$  is the Fourier transform of  $\Phi$  and  $\langle x, y \rangle = q(x+y) - q(x) - q(y)$  is the bilinear form on  $L$  associated to  $q$ . Fix the Haar measure  $d^{\times} x$  on  $L^{\times}$  to be the one defined by

$$d^{\times} x = \begin{cases} L(1, 1_v)^2 \frac{dx}{|q(x)|_v}, & \text{if } L = F_v^2; \\ L(1, 1_L) \frac{dx}{|q(x)|_v}, & \text{if } L \text{ is a quadratic field extension over } F_v; \\ L(1, 1_v) \frac{dx}{|q(x)|_v^2}, & \text{if } L \text{ is a quaternion algebra.} \end{cases}$$



Endow  $L^\times/F_v^\times$  with the quotient Haar measure. Let  $K$  be a quadratic field extension over  $F$  and  $B$  a quaternion algebra over  $F$ . For local Haar measures on  $K_v^\times/F_v^\times$  and  $B_v^\times/F_v^\times$ , their product Haar measures on  $K_\mathbb{A}^\times/\mathbb{A}^\times$  and  $B_\mathbb{A}^\times/\mathbb{A}^\times$  satisfy the following:

$$\text{Vol}(K^\times \backslash K_\mathbb{A}^\times / \mathbb{A}^\times) = 2L(1, \eta), \quad \text{Vol}(B^\times \backslash B_\mathbb{A}^\times / \mathbb{A}^\times) = 2.$$

Thus these measures can be taken as the ones used in the above statement of Waldspurger's formula. From now on, we always use these measures and the additive character  $\psi$  on  $\mathbb{A}$ .

**2.1. Petersson Pairing Formula.** Let  $\sigma$  be a cuspidal automorphic representation of  $\text{GL}_2(\mathbb{A})$  and  $\tilde{\sigma}$  its contragredient. Let  $N$  be the unipotent subgroup  $N = \left\{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, x \in F \right\}$  of  $\text{GL}_2$ . View  $\psi$  as a character on  $N(F) \backslash N(\mathbb{A})$  and the Haar measure  $da$  on  $\mathbb{A}$  as the one on  $N(\mathbb{A})$ . For any  $\phi \in \sigma$ , let  $W_\phi \in \mathcal{W}(\sigma, \psi)$  be the Whittaker function associated to  $\phi$ :

$$W_\phi(g) := \int_{N(F) \backslash N(\mathbb{A})} \phi(n g) \overline{\psi(n)} dn.$$

Recall there is a  $\text{GL}_2(F_v)$ -pairing on  $\mathcal{W}_{\sigma_v, \psi_v} \times \mathcal{W}_{\tilde{\sigma}_v, \psi_v^{-1}}$ : for any local Whittaker functions  $W_{1,v} \in \mathcal{W}(\sigma_v, \psi_v), W_{2,v} \in \mathcal{W}(\tilde{\sigma}_v, \psi_v^{-1})$ ,

$$\langle W_{1,v}, W_{2,v} \rangle = \int_{F_v^\times} W_{1,v} \begin{pmatrix} a & \\ & 1 \end{pmatrix} W_{2,v} \begin{pmatrix} a & \\ & 1 \end{pmatrix} d^\times a.$$

Define the Petersson pairing on  $\sigma \times \tilde{\sigma}$  by:

$$\langle \phi_1, \phi_2 \rangle_{\text{Pet}} := \int_{Z(\mathbb{A}) \text{GL}_2(F) \backslash \text{GL}_2(\mathbb{A})} \phi_1(g) \phi_2(g) dg, \quad \phi_1 \in \sigma, \phi_2 \in \tilde{\sigma},$$

where  $Z \cong F^\times$  is the center of  $\text{GL}_2$ .

**Proposition 2.1.** *For any  $\phi_1 \in \sigma, \phi_2 \in \tilde{\sigma}$  pure tensors and write  $W_{\phi_i} = \otimes_v W_{i,v}$ ,  $i = 1, 2$ , we have*

$$(2.2) \quad \langle \phi_1, \phi_2 \rangle_{\text{Pet}} = 2L(1, \sigma, \text{ad}) L(2, 1_F)^{-1} \prod_v \alpha_v(W_{1,v}, W_{2,v}),$$

where for any place  $v$  of  $F$

$$\alpha_v(W_{1,v}, W_{2,v}) = \frac{1}{L(1, \sigma_v, \text{ad}) L(1, 1_v) L(2, 1_v)^{-1}} \cdot \langle W_{1,v}, W_{2,v} \rangle,$$

*Proof.* We may assume that the cuspidal automorphic representation  $\sigma$  is also unitary and identify  $\tilde{\sigma}$  with  $\bar{\sigma}$ . Let  $G = \text{GL}_2$  over  $F$ ,  $P$  the parabolic subgroup of upper triangle matrices in  $G$ , and let  $U = \prod_v U_v$  be a maximal compact subgroup of  $G(\mathbb{A})$ . For any place  $v$  of  $F$ , with respect to the Iwasawa decomposition of  $G(F_v)$ ,

$$g = a \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & b \end{pmatrix} k \in G(F_v), \quad a, b \in F_v^\times, x \in F_v, k \in U_v,$$

choose the measure  $dk$  on  $U_v$  such that  $dg = |b| dx d^\times a d^\times b dk$  is the fixed local Haar measure on  $G(F_v)$ . Note that for  $v$  non-archimedean,  $U_v$  has volume  $L(2, 1_v)^{-1} |\delta_v|^{1/2}$  with respect to  $dk$  and has volume  $L(2, 1_v)^{-1} |\delta_v|^2$  with respect to the fixed measure on  $G(F_v)$ ; for  $v$  archimedean,  $U_v$  has volume  $L(2, 1_v)^{-1}$  with respect to  $dk$ .

By Lemma 2.3. in [10], for any Bruhat-Schwartz function  $\Phi_v \in \mathcal{S}(F_v^2)$ , we have

$$\int_{F_v^\times \times U_v} \Phi([0, b]k) |b|^2 d^\times b dk = \widehat{\Phi}_v(0),$$

where  $\widehat{\Phi}_v$  is the Fourier transformation of  $\Phi_v$  and  $\widehat{\Phi}_v(0)$  is independent of the choice of the additive character  $\psi_v$ . For any  $\Phi \in \mathcal{S}(\mathbb{A}^2)$ , let

$$F(s, g, \Phi) = |\det g|^s \int_{\mathbb{A}^\times} \Phi([0, b]g) |b|^{2s} d^\times b,$$

and define the Eisenstein series

$$E(s, g, \Phi) := \sum_{\gamma \in P(F) \backslash G(F)} F(s, \gamma g, \Phi), \quad \text{Re}(s) \gg 0.$$

By Possion summation formula,

$$\begin{aligned}
E(s, g, \Phi) &= |\det g|^s \int_{F^\times \setminus \mathbb{A}^\times} \left( \sum_{\xi \in F^2 \setminus \{0\}} \Phi(a\xi g) \right) |a|^{2s} d^\times a \\
&= |\det g|^s \int_{|a| \geq 1} \left( \sum_{\xi \in F^2 \setminus \{0\}} \Phi(a\xi g) \right) |a|^{2s} d^\times a \\
&\quad + |\det g|^{s-1} \int_{|a| \geq 1} \left( \sum_{\xi \in F^2 \setminus \{0\}} \widehat{\Phi}(g^{-1}\xi^t a) \right) |a|^{2-2s} d^\times a \\
&\quad + |\det g|^{s-1} \widehat{\Phi}(0) \int_{|a| \leq 1} |a|^{2s-2} d^\times a \\
&\quad - |\det g|^s \Phi(0) \int_{|a| \leq 1} |a|^{2s} d^\times a.
\end{aligned}$$

It shows that  $E(s, g, \Phi)$  has meromorphic continuation to the whole  $s$ -plane, has only possible poles at  $s = 0$  and  $1$ , and its residue at  $s = 1$  is equal to

$$\text{Res}_{s=1} E(s, g, \Phi) = \widehat{\Phi}(0) \lim_{s \rightarrow 1} (s-1) \int_{|a| \leq 1} |a|^{2s-2} d^\times a = \frac{\widehat{\Phi}(0)}{2} \text{Res}_{s=1} L(s, 1_F),$$

which is independent of  $g$ . By unfolding the Eisenstein series and Fourier expansions of  $\phi_i$ , the integral

$$\begin{aligned}
Z(s, W_{\phi_1}, W_{\phi_2}, \Phi) &= Z(s, \phi_1, \phi_2, \Phi) := \int_{[Z \backslash G]} \phi_1(g) \phi_2(g) E(s, g, \Phi) dg \\
&= \int_{N(\mathbb{A}) \backslash G(\mathbb{A})} |\det g|^s W_{\phi_1}(g) W_{\phi_2}(g) \Phi([0, 1]g) dg
\end{aligned}$$

has an Euler product if  $\Phi \in S(\mathbb{A}^2)$  is a pure tensor. For each place  $v$  of  $F$  and  $\Phi_v \in S(F_v^2)$ , denote

$$Z(s, W_{1,v}, W_{2,v}, \Phi_v) = \int_{N(F_v) \backslash G(F_v)} |\det g|^s W_{1,v}(g) W_{2,v}(g) \Phi_v([0, 1]g) dg,$$

which has meromorphic continuation to the whole  $s$ -plane, and moreover, for  $v \nmid \infty$ , the fractional ideal of  $\mathbb{C}[q_v^s, q_v^{-s}]$  of all  $Z(s, W_{1,v}, W_{2,v}, \Phi_v)$  with  $W_{1,v} \in \mathcal{W}(\sigma_v, \psi_v)$ ,  $W_{2,v} \in \mathcal{W}(\tilde{\sigma}_v, \psi_v^{-1})$ , and  $\Phi_v \in \mathcal{S}(F_v^2)$  is generated by  $L(s, \sigma_v \times \tilde{\sigma}_v)$ . It is also known that for each  $v$ ,

$$Z(1, W_{1,v}, W_{2,v}, \Phi_v) = \int_{F^\times} W_{1,v} \begin{pmatrix} a & \\ & 1 \end{pmatrix} W_{2,v} \begin{pmatrix} a & \\ & 1 \end{pmatrix} d^\times a \cdot \iint_{F_v^\times \times U_v} \Phi_v([0, b]k) |b|^2 d^\times b dk,$$

with Haar measures chosen above. Let  $\Phi = \otimes_v \Phi_v \in \mathcal{S}(\mathbb{A}^2)$  be a pure tensor such that  $\widehat{\Phi}(0) \neq 0$  and take residue at  $s = 1$  on the two sides of

$$Z(s, \phi_1, \phi_2, \Phi) = \prod_v Z(s, W_{1,v}, W_{2,v}, \Phi_v),$$

we have

$$\langle \phi_1, \phi_2 \rangle_{\text{Pet}} \text{Res}_{s=1} E(s, g, \Phi) = \text{Res}_{s=1} L(s, \sigma \times \tilde{\sigma}) \widehat{\Phi}(0) \prod_v \frac{\langle W_{1,v}, W_{2,v} \rangle}{L(1, \sigma_v \times \tilde{\sigma}_v)},$$

or

$$\frac{L(1, \sigma, \text{ad})}{\langle \phi_1, \phi_2 \rangle_{\text{Pet}}} = \frac{1}{2} \prod_v \frac{L(1, \sigma_v, \text{ad}) L(1, 1_{F_v})}{\langle W_{1,v}, W_{2,v} \rangle}.$$

The formula in the Proposition follows. □

## 2.2. $U$ -level Pairing.

**Lemma 2.2.** *Let  $B$  be a quaternion algebra over a number field  $F$  and denote by  $r, s, t$  integers such that  $B \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{H}^r \times M_2(\mathbb{R})^s \times M_2(\mathbb{C})^t$ . For  $U \subset \hat{B}^\times$  an open compact subgroup, the volume of  $X_U$ , defined after Definition 1.7, is given by*

$$\text{Vol}(X_U) = 2(4\pi^2)^{-d} \#(\mathbb{A}_f^\times / F^\times U_Z) \cdot \frac{\text{Vol}(U_Z)}{\text{Vol}(U)}.$$

where  $U_Z = U \cap \widehat{F}^\times$  and the volumes  $\text{Vol}(U_Z)$  and  $\text{Vol}(U)$  are with respect to Tamagawa measure so that  $\text{Vol}(\text{GL}_2(\mathcal{O}_v)) = L(2, 1_v)^{-1} \text{Vol}(\mathcal{O}_v)^4$ , and  $\text{Vol}(B_v^\times) = L(2, 1_v)^{-1} \text{Vol}(\mathcal{O}_v)^4 (q_v - 1)^{-1}$  for  $B_v$  division. In particular, if  $U$  contains  $\widehat{\mathcal{O}}^\times$ , then

$$\text{Vol}(X_U) = 2(4\pi^2)^{-d} |D_F|^{-1/2} \cdot h_F \cdot \text{Vol}(U)^{-1},$$

where  $h_F$  is the class number of  $F$ .

*Proof.* (See also [40] for the case  $s = 1$  and  $t = 0$ ). Denote by  $B^1 := \{b \in B^\times | q(b) = 1\}$  with  $q$  the reduced norm on  $B$ . For each place  $v$  of  $F$ , we have the exact sequence

$$1 \longrightarrow B_v^1 \longrightarrow B_v^\times \longrightarrow q(B_v^\times) \longrightarrow 1,$$

and define the Haar measure  $dh_v$  on  $B_v^1$  such that it makes the Haar measure on  $q(B_v^\times)$ , obtained by the restriction of the Haar measure on  $B_v^\times$ , equal to the quotient of the Haar measure on  $B_v^\times$  by  $dh_v$ . The product of these local measures give the Tamagawa measure on  $B_\mathbb{A}^1$  so that  $\text{Vol}(B^1 \backslash B_\mathbb{A}^1) = 1$ . This follows from the fact that the Tamagawa numbers of  $B^1$  and  $B^\times$  are 1 and 2, respectively. Assume that  $B \otimes_\mathbb{Q} \mathbb{R} = \mathbb{H}^r \times M_2(\mathbb{R})^s \times M_2(\mathbb{C})^t$ . We assume that  $s + t > 0$  first and let  $\Sigma \subset \infty$  be the subset of infinite places of  $F$  where  $B$  splits. By the strong approximation theorem,  $B_\mathbb{A}^1 = B^1 B_\infty^1 U^1$ , where  $U^1 = U \cap B_{\mathbb{A}_f}^1$  is an open compact subgroup of  $B_{\mathbb{A}_f}^1$ . It follows that

$$B^1 \backslash B_\mathbb{A}^1 = B^1 \backslash B^1 B_\infty^1 U^1 = (\Gamma \backslash B_\Sigma^1) B_\infty^{1, \Sigma} U^1$$

where  $\Gamma = B^1 \cap U^1$  and we identify  $\Gamma \backslash B_\Sigma^1$  with the fundamental domain of this quotient.

For a real place  $v$  of  $F$ ,  $B_v^1 \cong \text{SL}_2(\mathbb{R})$ . By the Iwasawa decomposition, any element is uniquely of the form

$$\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \quad x \in \mathbb{R}, y \in \mathbb{R}_+, \theta \in [0, 2\pi).$$

The measure on  $B_v^1$  is  $dx dy d\theta / 2y^2$  with  $dx dy$  the usual Lebesgue and  $\theta$  has volume  $2\pi$ . For a complex place  $v$  of  $F$ ,  $B_v^1 \cong \text{SL}_2(\mathbb{C})$ . By the Iwasawa decomposition, any element in  $\text{SL}_2(\mathbb{C})$  is uniquely of form

$$\begin{pmatrix} 1 & z \\ & 1 \end{pmatrix} \begin{pmatrix} v^{1/2} & \\ & v^{-1/2} \end{pmatrix} u, \quad z \in \mathbb{C}, v \in \mathbb{R}_+, u \in \text{SU}_2,$$

The measure on  $B_v^1$  is  $dx dy dv du / v^3$  with  $z = x + yi$ ,  $dx, dy, dv$  the usual Lebesgue measure, and  $du$  has volume  $8\pi^2$  (see [37]). It follows that

$$\text{Vol}(\Gamma \backslash B_\Sigma^1) = 2^{-t} (4\pi^2)^{s+2t} w_U^{-1} \cdot \text{Vol} \left( \Gamma \backslash (\mathcal{H}_2^s \times \mathcal{H}_3^t), \wedge \frac{dx dy}{4\pi y^2} \wedge \frac{dx dy dv}{\pi^2 v^3} \right),$$

where  $w_U = \#\{\pm 1\} \cap U$ . On the other hand, for any infinite place  $v \notin \Sigma$ ,  $\text{Vol}(B_v^1) = 4\pi^2$ . It follows that

$$w_U^{-1} \cdot 2^{-t} (4\pi^2)^d \cdot \text{Vol} \left( \Gamma \backslash (\mathcal{H}_2^s \times \mathcal{H}_3^t), \wedge \frac{dx dy}{4\pi y^2} \wedge \frac{dx dy dv}{\pi^2 v^3} \right) \cdot \text{Vol}(U^1) = 1,$$

where  $d = [F : \mathbb{Q}]$ . Let  $B_+^\times \subset B^\times$  be the subgroup of elements whose norms are positive at all real places. Now consider the natural map

$$(B^1 \cap U^1) \backslash (\mathcal{H}_2^s \times \mathcal{H}_3^t) \longrightarrow (B_+^\times \cap U) \backslash (\mathcal{H}_2^s \times \mathcal{H}_3^t),$$

whose degree is just

$$[(B_+^\times \cap U) : (B^1 \cap U^1) \mu_U] = [\det(B_+^\times \cap U) : \mu_U^2] = [\mu_U' : \mu_U^2].$$

Here  $\mu_U = F^\times \cap U$  and  $\mu_U' = F_+^\times \cap \det U$ , subgroups of  $\mathcal{O}_F^\times$  with finite index. It follows that

$$\begin{aligned} \text{Vol}(X_U) &= \text{Vol}((B_+^\times \cap U) \backslash (\mathcal{H}_2^s \times \mathcal{H}_3^t) \cdot \#(F_+^\times \backslash \widehat{F}^\times / \det U)) \\ &= \frac{2^t w_U}{(4\pi^2)^d \cdot \text{Vol}(U^1) \cdot [\mu_U' : \mu_U^2]} \cdot \#(F_+^\times \backslash \widehat{F}^\times / \det U). \end{aligned}$$

Note that

$$\frac{\#(\widehat{F}^\times / F_+^\times \det(U))}{\#(\widehat{F}^\times \backslash F^\times U_Z)} = [F^\times U_Z : F_+^\times \det U] = [F^\times : F_+^\times] \frac{\text{Vol}(U_Z)}{\text{Vol}(\det(U))} [\mu_U' : \mu_U].$$

Note that  $[F^\times : F_+^\times] = 2^{r+s}$ ,  $[\mu_U : \mu_U^2] = 2^{r+s+t-1} w_U$ , and  $\text{Vol}(U) = \text{Vol}(U^1) \text{Vol}(\det(U))$ , we have

$$\text{Vol}(X_U) = 2(4\pi^2)^{-d} \#(\widehat{F}^\times / F^\times U_Z) \cdot \frac{\text{Vol}(U_Z)}{\text{Vol}(U)}.$$

Now assume that  $s = t = 0$ . Note that the Tamagawa number of  $B^\times$  is 2,  $\text{Vol}(B_v^\times/F_v^\times) = 4\pi^2$  for any  $v|\infty$ , and the decomposition

$$B^\times \mathbb{A}^\times \backslash B_\mathbb{A}^\times = F_\infty^\times \backslash B_\infty^\times \times B^\times \widehat{F}^\times \backslash \widehat{B}^\times.$$

It follows that  $\text{Vol}(B^\times \widehat{F}^\times \backslash \widehat{B}^\times) = 2(4\pi^2)^{-d}$ . Let  $\gamma_1, \dots, \gamma_h$  be a complete set of representatives in  $\widehat{B}^\times$  of the coset  $B^\times \backslash \widehat{B}^\times / U$ . Consider the natural map

$$B^\times \backslash B^\times \gamma_i U \longrightarrow B^\times \widehat{F}^\times \backslash B^\times \widehat{F}^\times \gamma_i U,$$

whose degree is  $\#\widehat{F}^\times / F^\times U_Z$ . Since

$$\text{Vol}(B^\times \widehat{F}^\times \backslash B^\times \widehat{F}^\times \gamma_i U) = \text{Vol}\left(\frac{\gamma_i(U/U_Z)\gamma_i^{-1}}{(B^\times \cap \gamma_i U \gamma_i^{-1})/\mu_Z}\right) = \frac{\text{Vol}(U)/\text{Vol}(U_Z)}{\#(B^\times \cap \gamma_i U \gamma_i^{-1})/\mu_Z}.$$

Thus

$$2(4\pi^2)^{-d} = \text{Vol}(B^\times \widehat{F}^\times \backslash \widehat{B}^\times) = (\#\widehat{F}^\times / F^\times U_Z)^{-1} \cdot \frac{\text{Vol}(U)}{\text{Vol}(U_Z)} \cdot \sum_{i=1}^h \frac{1}{\#(B^\times \cap \gamma_i U \gamma_i^{-1})/\mu_Z}.$$

□

**2.3.  $c_1$ -level Periods.** Now take  $f_1 \in V(\pi, \chi), f_2 \in V(\widetilde{\pi}, \chi^{-1})$  to be non-zero test vectors defined before. Let  $\sigma = \pi^{\text{JL}}$ , take  $\phi_1 \in \sigma$  and  $\phi_2 \in \widetilde{\sigma}$  to be normalized new vectors. The  $c_1$ -level periods  $P_\chi^0(f_1), P_{\chi^{-1}}^0(f_2)$  are related to the periods in Waldspurger's formula by the following lemma:

**Lemma 2.3.** *Let  $b \subset \mathcal{O}$  be a non-zero ideal of  $F$  and denote by  $\text{Pic}_{K/F}(\mathcal{O}_b)$  the group  $\widehat{K}^\times / K^\times \widehat{F}^\times \widehat{\mathcal{O}}_b^\times$ . Then there is a relative class number formula:*

$$L^{(b)}(1, \eta) \cdot \|D_{K/F} b^2 \delta\|^{1/2} \cdot 2^{-r_{K/F}} = \frac{\#\text{Pic}_{K/F}(\mathcal{O}_b) \cdot R_b}{\#\kappa_b \cdot w_b},$$

where  $r_{K/F} = \text{rank} \mathcal{O}_K^\times - \text{rank} \mathcal{O}^\times$ ,  $w_b = [\mathcal{O}_{b, \text{tor}}^\times : \mathcal{O}_{\text{tor}}^\times]$ ,  $R_b$  is the quotient of the regulator of  $\mathcal{O}_b^\times$  by the one of  $\mathcal{O}^\times$ , and  $\kappa_b$  is the kernel of the natural morphism from  $\text{Pic}(\mathcal{O})$  to  $\text{Pic}(\mathcal{O}_b)$ . Define a constant  $\nu_b := 2^{-r_{K/F}} R_b^{-1} \cdot \#\kappa_b w_b$ . Then it follows that

$$P_\chi(f) = 2L_{c_1}(1, \eta) \|D_{c_1}^2 \delta\|^{-1/2} \nu_{c_1}^{-1} \cdot P_\chi^0(f).$$

*Proof.* There are exact sequences

$$1 \rightarrow \kappa_b \rightarrow \widehat{F}^\times / F^\times \widehat{\mathcal{O}}_F^\times \rightarrow \widehat{K}^\times / K^\times \widehat{\mathcal{O}}_b^\times \rightarrow \widehat{K}^\times / K^\times \widehat{F}^\times \widehat{\mathcal{O}}_b^\times \rightarrow 1,$$

and

$$1 \rightarrow \mathcal{O}_K^\times / \mathcal{O}_b^\times \rightarrow \widehat{\mathcal{O}}_K^\times / \widehat{\mathcal{O}}_b^\times \rightarrow \widehat{K}^\times / K^\times \widehat{\mathcal{O}}_b^\times \rightarrow \widehat{K}^\times / K^\times \widehat{\mathcal{O}}_K^\times \rightarrow 1.$$

It follows that

$$\#\text{Pic}_{K/F}(\mathcal{O}_b) = \#\widehat{K}^\times / K^\times \widehat{F}^\times \widehat{\mathcal{O}}_b^\times = \frac{h_K}{h_F} \cdot [\widehat{\mathcal{O}}_K^\times : \widehat{\mathcal{O}}_b^\times] \cdot [\mathcal{O}_K^\times : \mathcal{O}_b^\times]^{-1} \cdot \#\kappa_b,$$

where  $h_K = \#\widehat{K}^\times / K^\times \widehat{\mathcal{O}}_K^\times$  is the ideal class number of  $K$  and similarly for  $h_F$ . By class number formula for  $F$  and  $K$ , we have

$$\text{Res}_{s=1} L(s, 1_F) = 2^{r_F+1} \frac{R_F h_F}{w_F \sqrt{|D_F|}}, \quad \text{Res}_{s=1} L(s, 1_K) = 2^{r_K+1} \frac{R_K h_K}{w_K \sqrt{|D_K|}}$$

where  $r_F = \text{rank} \mathcal{O}_F^\times$ ,  $D_F$  is the discriminant of  $F$ ,  $R_F$  is the regulator of  $\mathcal{O}^\times$ ,  $h_F$  the ideal class number of  $F$ ,  $w_F = \#\mathcal{O}_{\text{tor}}^\times$ , and similar for  $r_K, D_K, R_K, h_K$  and  $w_K$ . Noting that  $|D_K|/|D_F| = |D_{K/F} \delta|_\mathbb{A}^{-1}$  and  $[\widehat{\mathcal{O}}_K^\times : \widehat{\mathcal{O}}_b^\times]^{-1} = L_b(1, \eta)|b|$ , we have that

$$\begin{aligned} L(1, \eta) &= \frac{h_K}{h_F} \cdot 2^{r_{K/F}} \frac{R_K w_K^{-1}}{R_F w_F^{-1}} \cdot \|D_{K/F} \delta\|^{-1/2} \\ &= \#\text{Pic}_{K/F}(\mathcal{O}_b) \cdot L_b(1, \eta) \cdot 2^{r_{K/F}} \cdot [\mathcal{O}_K^\times : \mathcal{O}_b^\times] \frac{R_K w_K^{-1}}{R_F w_F^{-1}} (\#\kappa_b)^{-1} \cdot \|D_{K/F} b^2 \delta\|^{-1/2}. \end{aligned}$$

The relative class number formula then follows. □

Let  $N$  be the conductor of  $\sigma = \pi^{\text{JL}}$ , let  $U \subset \widehat{B}^\times$  be an open compact subgroup, recall

$$\langle f_1, f_2 \rangle_U = \frac{\langle f_1, f_2 \rangle_{\text{Pet}}}{2} \text{Vol}(X_U), \quad \langle \phi_1, \phi_2 \rangle_{U_0(N)} = \frac{\langle \phi_1, \phi_2 \rangle_{\text{Pet}}}{2} \text{Vol}(X_{U_0(N)}).$$

Applying Proposition 2.1, Lemma 2.2, and Lemma 2.3, Waldspurger's formula (2.1) implies the following.

**Proposition 2.4.** *Let  $U = \prod_v U_v \subset \widehat{B}^\times$  be an open compact subgroup such that  $\widehat{\mathcal{O}}^\times \subset U$ . Denote by  $\gamma_v = \text{Vol}(U_0(N)_v)^{-1} \text{Vol}(U_v)$  for all finite places  $v$  and  $\gamma_v = 1$  for  $v|\infty$ . Let  $\phi_1 \in \pi^{\text{JL}}, \phi_2 \in \widetilde{\pi}^{\text{JL}}$  be any forms with  $\langle \phi_1, \phi_2 \rangle_{U_0(N)} \neq 0$  and let  $\alpha(W_{1,v}, W_{2,v})$  be corresponding local constants defined in Proposition 2.1. Let  $f_1 \in \pi, f_2 \in \widetilde{\pi}$  be any pure tensors with  $(f_1, f_2)_{\text{Pet}} \neq 0$  and  $\beta(f_{1,v}, f_{2,v})$  corresponding constants defined in (2.1). Then we have*

$$(2.3) \quad (2L_{c_1}(1, \eta) |Dc_1^2 \delta|_{\mathbb{A}}^{1/2} \nu_{c_1}^{-1})^2 \cdot \frac{P_\chi^0(f_1) P_{\chi^{-1}}^0(f_2)}{\langle f_1, f_2 \rangle_U} = \frac{L(1/2, \pi, \chi)}{\langle \phi_1, \phi_2 \rangle_{U_0(N)}} \cdot \prod_v \alpha_v(W_{1,v}, W_{2,v}) \beta_v(f_{1,v}, f_{2,v}) \gamma_v,$$

where  $\nu_{c_1}$  is defined as in Lemma 2.3.

It is now clear that the explicit Waldspurger formula will follow from the computation of these local factors. In the next section, we will choose  $\phi_1, \phi_2$  to be normalized new vectors in  $\pi^{\text{JL}}$  and  $\widetilde{\pi}^{\text{JL}}$ , respectively, choose non-zero  $f_1 \in V(\pi, \chi), f_2 \in V(\widetilde{\pi}, \chi)$ , and compute the related local factors in (2.3).

We obtain Explicit Gross-Zagier formula from Yuan-Zhang-Zhang's formula via a similar way. Let  $F$  be a totally real field and  $X$  a Shimura curve over  $F$  associated to an incoherent quaternion algebra  $\mathbb{B}$ . Let  $A$  be an abelian variety over  $F$  parametrized by  $X$  and let  $\pi_A = \text{Hom}_\xi^0(X, A)$  be the associated automorphic representation of  $\mathbb{B}^\times$  over the field  $M := \text{End}^0(A)$  and  $\omega$  its central character. Let  $K$  be a totally imaginary quadratic extension over  $F$  and  $\chi : K_{\mathbb{A}}^\times \rightarrow L^\times$  a finite order Hecke character over a finite extension  $L$  of  $M$  such that  $\omega \cdot \chi|_{\mathbb{A}^\times} = 1$  and for all places  $v$  of  $F$ ,  $\epsilon(1/2, \pi_A, \chi) = \chi_v \eta_v(-1) \epsilon(\mathbb{B}_v)$ . Fix an embedding  $K_{\mathbb{A}} \rightarrow \mathbb{B}$  and  $K_{\mathbb{A}}^\times \rightarrow \mathbb{B}^\times$ , let  $P \in X^{K^\times}(K^{\text{ab}})$  and define

$$P_\chi(f) = \int_{K_{\mathbb{A}}^\times / K^\times \mathbb{A}^\times} f(P)^{\sigma_t} \otimes_M \chi(t) dt \in A(K^{\text{ab}})_{\mathbb{Q}} \otimes_M L,$$

where we use the Haar measure so that the totally volume of  $K_{\mathbb{A}}^\times / K^\times \mathbb{A}^\times$  is  $2L(1, \eta)$ , and  $\eta$  is the quadratic Hecke character on  $\mathbb{A}^\times$  associated to the extension  $K/F$ . We further assume for all non-archimedean places  $v$  that the compact subgroup  $\mathcal{O}_{K_v}^\times / \mathcal{O}_v^\times$  has a volume in  $\mathbb{Q}^\times$  and fix a local invariant pairing  $(\ , \ )_v$  on  $\pi_{A,v} \times \pi_{A^\vee,v}$  valued in  $M$ . Define  $\beta(f_{1,v}, f_{2,v}) \in L$ , with  $(f_{1,v}, f_{2,v})_v \neq 0$ , by

$$\beta(f_{1,v}, f_{2,v}) = \frac{L(1, \eta_v) L(1, \pi_v, \text{ad})}{L(1/2, \pi_v, \chi_v) L(2, 1_{F_v})} \int_{K_v^\times / F_v^\times} \frac{(\pi(t_v) f_{1,v}, f_{2,v})_v}{(f_{1,v}, f_{2,v})_v} \chi(t_v) dt_v \in L.$$

where we take an embedding of  $L$  into  $\mathbb{C}$  and the above integral lies in fact in  $L$  and does not depend on the embedding.

Then for any pure tensors  $f_1 \in \pi_A, f_2 \in \pi_{A^\vee}$  with  $(f_1, f_2) \neq 0$ , Yuan-Zhang-Zhang obtained the following celebrated formula in [40] as an identity in  $L \otimes_{\mathbb{Q}} \mathbb{C}$ :

$$(2.4) \quad \frac{\langle P_\chi(f_1), P_{\chi^{-1}}(f_2) \rangle_{K,L}}{\text{Vol}(X_U)^{-1} (f_1, f_2)_U} = \frac{L'(1/2, \pi_A, \chi)}{L(1, \pi_A, \text{ad}) L(2, 1_F)^{-1}} \prod_v \beta_v(f_{1,v}, f_{2,v}).$$

Note that we use height over  $K$  and the one used in [40] is over  $F$ , the Haar measure to define  $P_\chi(f)$  is different from the one in [40] by  $2L(1, \eta)$ , and the measure to define  $\text{Vol}(X_U)$  is different from the one in [40] by 2. Similar to Proposition 2.4, we have now

**Proposition 2.5.** *Let  $U = \prod_v U_v \subset \widehat{B}^\times$  be a pure product open compact subgroup such that  $\widehat{\mathcal{O}}^\times \subset U$ . Denote by  $\gamma_v = \text{Vol}(U_0(N)_v) \text{Vol}(U_v)^{-1}$  for all finite places  $v$  and  $\gamma_v = 1$  for  $v|\infty$ . Let  $\phi \in \pi_A^{\text{JL}}$  be any nonzero form and let  $\alpha(W_v, \bar{W}_v)$  be corresponding local constants defined in Proposition 2.1. Let  $f_1 \in \pi_A, f_2 \in \pi_{A^\vee}$  be any pure tensors with  $(f_1, f_2) \neq 0$  and  $\beta(f_{1,v}, f_{2,v})$  corresponding constants defined in (2.4). Then we have*

$$(2.5) \quad (2L_{c_1}(1, \eta) |Dc_1^2 \delta|_{\mathbb{A}}^{1/2} \nu_{c_1}^{-1})^2 \cdot \frac{\langle P_\chi^0(f_1), P_{\chi^{-1}}^0(f_2) \rangle_{K,L}}{(f_1, f_2)_U} = \frac{L'(1/2, \pi_A, \chi)}{\langle \phi, \phi \rangle_{U_0(N)}} \prod_v \alpha_v(W_{1,v}, W_{2,v}) \beta_v(f_{1,v}, f_{2,v}) \gamma_v.$$

We will study the local factors appearing in formulas in Propositions 2.4 and 2.5 in the next section.



**2.4. Proof of Main Results.** In this subsection, we prove Theorems 1.5, 1.6, 1.8, 1.9 and 1.10, assuming local results proved in section 3.

**Proof of Theorem 1.8.** We first give a proof of the explicit Waldspurger formula. In the equation (2.3), take non-zero  $f_1 \in V(\pi, \chi)$ ,  $f_2 \in V(\tilde{\pi}, \chi^{-1})$ , and  $\phi_1^0$  (resp.  $\phi_2^0$ ) the normalized new vector of  $\pi^{\text{JL}}$  (resp.  $\tilde{\pi}^{\text{JL}}$ ). Let  $W_{\phi_i^0} := W_i \otimes_v W_{i,v}$  be corresponding Whittaker functions of  $\phi_i^0$ ,  $i = 1, 2$ . Let  $R \subset B$  be the order as defined in Theorem 1.8 and  $U = \hat{R}^\times$ . Denote

$$\alpha_v := \alpha_v(W_{1,v}, W_{2,v}) \cdot |\delta|_v^{1/2}, \quad \beta_v := \beta_v(f_{1,v}, f_{2,v}) \cdot |D\delta|_v^{-1/2}.$$

Then the equation (2.3) becomes

$$4|Dc_1^2\delta^2|_{\mathbb{A}}^{1/2}\nu_{c_1}^{-2}\frac{P_{\chi}^0(f_1)P_{\chi^{-1}}^0(f_2)}{\langle f_1, f_2 \rangle_U} = \frac{L^{(\Sigma)}(1/2, \pi, \chi)}{\langle \phi_1^0, \phi_2^0 \rangle_{U_0(N)}} L_{\Sigma}(1/2, \pi, \chi) L_{c_1}(1, \eta)^{-2} |c_1|_{\mathbb{A}}^{-1} \prod_v \alpha_v \beta_v \gamma_v.$$

Let  $\Sigma$  be the set in Theorem 1.8 and  $\Sigma_{\infty} = \Sigma \cap \infty$  and  $\Sigma_f = \Sigma \setminus \Sigma_{\infty}$ . Comparing with the formula (2.3), the proof of the explicit formula in Theorem 1.8 is reduced to showing that

$$L_{\Sigma_f}(1/2, \pi, \chi) L_{c_1}(1, \eta)^{-2} |c_1|_{\mathbb{A}}^{-1} \prod_{v \nmid \infty} \alpha_v \beta_v \gamma_v = 2^{\#\Sigma_D}$$

and

$$L_{\Sigma_{\infty}}(1/2, \pi, \chi) \prod_{v|\infty} \alpha_v \beta_v \gamma_v = C_{\infty}^{-1},$$

which are given by Lemma 3.13 and Lemma 3.14.

**Proof of Theorem 1.10.** In the situation of Theorem 1.10, identify  $\tilde{\pi}$  with  $\bar{\pi}$ , by Theorem 1.8 we have

$$L^{(\Sigma)}(1/2, \pi, \chi) = 2^{-\#\Sigma_D+2} (4\pi^3)^d \frac{\langle \phi^0, \bar{\phi}^0 \rangle_{U_0(N)}}{\nu_{c_1}^2 \sqrt{|D_K|} \|c_1^2\|} \frac{|P_{\chi}^0(f)|^2}{\langle f_1, f_2 \rangle_U}.$$

The formula in Theorem 1.10 follows by noting the following facts:

- (i)  $\nu_{c_1} = 2^{1-d} u_1$ ,
- (ii)  $\langle \phi^0, \bar{\phi}^0 \rangle_{U_0(N)} = (2\pi)^{-d} \langle \phi, \phi \rangle_{U_0(N)}$ , where  $\phi$  is the Hilbert new form of  $\pi_A^{\text{JL}}$ . Here the last fact is obtained by applying the formula in Proposition 2.1 to  $\phi$  and  $\phi^0$  and the comparison of local Whittaker pairings at infinity, see remark before Proposition 3.12.
- (iii) let  $g_1, \dots, g_n \in \hat{B}^\times$  be a complete set of representatives of the coset  $X = B^\times \backslash \hat{B}^\times / \hat{R}^\times$  and let  $w_i = \#(B^\times \cap g_i \hat{R}^\times g_i^{-1} / \mathcal{O}^\times)$ , then as in the proof of lemma 2.2, for  $U = \hat{R}^\times$

$$\langle f, \bar{f} \rangle_U = 2^{-1} \text{Vol}(X_U) \langle f, \bar{f} \rangle_{\text{Pet}} = \sum_{i=1}^n |f(g_i)|^2 w_i^{-1} = \langle \sum f(g_i) w_i^{-1} [g_i], \sum f(g_i) w_i^{-1} [g_i] \rangle = \langle f, f \rangle,$$

where we identify  $f$  with its image under the map  $V(\pi, \chi) \longrightarrow \mathbb{C}[X]$  and  $\langle \cdot, \cdot \rangle$  is the height pairing on  $\mathbb{C}[X]$ .

**Proof of Theorem 1.5.** To show the explicit Gross-Zagier formula in Theorem 1.5, similarly as above, we apply formula (2.5) in Proposition 2.5, to non-zero forms  $f_1 \in V(\pi_A, \chi)$  and  $f_2 \in V(\pi_{A^\vee}, \chi^{-1})$ ,  $\phi^0$  the normalized new vector of  $\pi_A^{\text{JL}}$ , and  $U = \mathcal{R}^\times$  as in Theorem 1.5. By Lemma 3.13 and Lemma 3.14, we have

$$L'^{(\Sigma)}(1/2, \pi, \chi) = 2^{-\#\Sigma_D+2} (4\pi^3)^d \frac{\langle \phi^0, \bar{\phi}^0 \rangle_{U_0(N)}}{\nu_{c_1}^2 \sqrt{|D_K|} \|c_1^2\|} \frac{\langle P_{\chi}^0(f_1), P_{\chi^{-1}}^0(f_2) \rangle_{K,L}}{(f_1, f_2)_U}.$$

Then the explicit Gross-Zagier formula follows again by noting the facts (i) and (ii) above.

**Proof of Theorem 1.9 and 1.6.** We now show that the variations of Explicit Waldspurger formula in Theorem 1.9 follows from the Waldspurger formula (2.1) and its explicit form in Theorem 1.8. Similarly for variation of explicit Gross-Zagier formula in Theorem 1.6.

Let  $f'_1 = \otimes_v f'_{1,v} \in \pi$ ,  $f'_2 = \otimes_v f'_{2,v} \in \tilde{\pi}$  be forms different from the test vectors  $f_1 = \otimes_v f_{1,v} \in V(\pi, \chi)$ ,  $f_2 = \otimes_v f_{2,v} \in V(\tilde{\pi}, \chi^{-1})$  from a finite set  $S$  of places of  $F$ , respectively, such that  $\langle f'_{1,v}, f'_{2,v} \rangle_v \neq 0$  and  $\beta(f'_{1,v}, f'_{2,v}) \neq 0$  for any  $v \in S$ . By Waldspurger formula (2.1), we have the following formulas

$$\frac{P_{\chi}^0(f_1) \cdot P_{\chi^{-1}}^0(f_2)}{\langle f_1, f_2 \rangle_U} = \mathcal{L}(\pi, \chi) \prod_v \beta(f_{1,v}, f_{2,v}), \quad \frac{P_{\chi}^0(f'_1) \cdot P_{\chi^{-1}}^0(f'_2)}{\langle f'_1, f'_2 \rangle_U} = \mathcal{L}(\pi, \chi) \prod_v \beta(f'_{1,v}, f'_{2,v}),$$

where

$$\mathcal{L}(\pi, \chi) = \left( \frac{\#\text{Pic}_{K/F}(\mathcal{O}_{c_1})}{2L(1, \eta)} \right)^2 \cdot \frac{2}{\text{Vol}(X_U)} \cdot \frac{L(1/2, \pi, \chi)}{2L(1, \pi, \text{ad})L(2, 1_F)^{-1}}.$$

It follows that

$$\frac{P_\chi^0(f_1) \cdot P_{\chi^{-1}}^0(f_2)}{\langle f_1, f_2 \rangle_U} = \frac{P_\chi^0(f'_1) \cdot P_{\chi^{-1}}^0(f'_2)}{\langle f'_1, f'_2 \rangle_U} \cdot \prod_{v \in S} \frac{\beta(f_{1,v}, f_{2,v})}{\beta(f'_{1,v}, f'_{2,v})}.$$

The variation formula follows immediately.

### 3. LOCAL THEORY

**Notations.** In this section, we denote by  $F$  a local field of characteristic zero, i.e. a finite field extension of  $\mathbb{Q}_v$  for some place  $v$  of  $\mathbb{Q}$ . Denote by  $|\cdot|$  the absolute value of  $F$  such that  $d(ax) = |a|dx$  for a Haar measure  $dx$  on  $F$ . Take an element  $\delta \in F^\times$  such that  $\delta\mathcal{O}$  is the different of  $F$  over  $\mathbb{Q}_v$  for  $v$  finite and  $\delta = 1$  for  $v$  infinite. For  $F$  non-archimedean, denote by  $\mathcal{O}$  the ring of integers in  $F$ ,  $\varpi$  a uniformizer,  $\mathfrak{p}$  its maximal ideal, and  $q$  the cardinality of its residue field. Let  $v : F \rightarrow \mathbb{Z} \cup \{\infty\}$  be the additive valuation on  $F$  such that  $v(\varpi) = 1$ . For  $\mu$  a (continuous) character on  $F^\times$ , denote by  $n(\mu)$  the conductor of  $\mu$ , that is, the minimal non-negative integer  $n$  such that  $\mu$  is trivial on  $(1 + \varpi^n \mathcal{O}) \cap \mathcal{O}^\times$ . We will always use the additive character  $\psi$  on  $F$  and the Haar measure  $da$  on  $F$  as in section 2 so that  $da$  is self dual to  $\psi$ .

Denote by  $K$  a separable quadratic extension of  $F$  and for any  $t \in K$ , write  $t \mapsto \bar{t}$  for the non-trivial automorphism of  $K$  over  $F$ . We use similar notations as those for  $F$  with a subscript  $K$ . If  $F$  is non-archimedean and  $K$  is non-split, denote by  $e$  the ramification index of  $K/F$ . Denote by  $\text{tr}_{K/F}$  (resp.  $N_{K/F}$ ) the trace (resp. norm) map from  $K$  to  $F$  and let  $D \in \mathcal{O}$  be an element such that  $D\mathcal{O}$  is the relative discriminant of  $K$  over  $F$ . For an integer  $c \geq 0$ , denote  $\mathcal{O}_c$  the order  $\mathcal{O} + \varpi^c \mathcal{O}_K$  in  $K$ . Let  $\eta : F^\times \rightarrow \{\pm 1\}$  be the character associated to the extension  $K$  over  $F$ . Let  $B$  be a quaternion algebra over  $F$ . Let  $\epsilon(B) = +1$  and  $\delta(B) = 0$  if  $B \cong M_2(F)$  is split, and  $\epsilon(B) = -1$  and  $\delta(B) = 1$  if  $B$  is division. Denote by  $G$  the algebraic group  $B^\times$  over  $F$  and we also write  $G$  for  $G(F)$ . We take the Haar measure on  $F^\times, K^\times$  and  $K^\times/F^\times$  as in section 2. In particular,  $\text{Vol}(\mathcal{O}^\times, d^\times a) = \text{Vol}(\mathcal{O}, da) = |\delta|^{1/2}$  and

$$\text{Vol}(K^\times/F^\times) = \begin{cases} 2, & \text{if } F = \mathbb{R} \text{ and } K = \mathbb{C}; \\ |\delta|^{1/2}, & \text{if } K \text{ is the unramified extension field of } F; \\ 2|D\delta|^{1/2}, & \text{if } K/F \text{ is ramified.} \end{cases}$$

For  $F$  non-archimedean and  $n$  a non-negative integer, define the following subgroups of  $\text{GL}_2(\mathcal{O})$ :

$$U_0(n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathcal{O}) \mid c \in \mathfrak{p}^n \right\}, \quad U_1(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_0(n) \mid d \in 1 + \varpi^n \mathcal{O} \right\}.$$

Let  $\pi$  be an irreducible admissible representation of  $G$  which is always assumed to be generic if  $G \cong \text{GL}_2$ . Denote by  $\omega$  the central character of  $\pi$  and  $\sigma = \pi^{\text{JL}}$  the Jacquet-Langlands correspondence of  $\pi$  to  $\text{GL}_2(F)$ . Let  $\chi$  be a character on  $K^\times$  such that

$$\chi|_{F^\times} \cdot \omega = 1.$$

For  $F$  non-archimedean, denote by

- $n$  - the conductor of  $\sigma$ , the minimal non-negative integer such that the invariant subspace  $\sigma^{U_1(n)}$  is nonzero.
- $c$  - the minimal non-negative integer  $c$  such that  $\chi$  is trivial on  $(1 + \varpi^c \mathcal{O}_K) \cap \mathcal{O}_K^\times$ .

Denote by

$$L(s, \pi, \chi) := L(s, \sigma \times \pi_\chi), \quad \epsilon(s, \pi, \chi) := \epsilon(s, \sigma \times \pi_\chi, \psi)$$

the Rankin-Selberg  $L$ -factor and  $\epsilon$ -factor of  $\sigma \times \pi_\chi$ , where  $\pi_\chi$  is the representation on  $\text{GL}_2(F)$  constructed from  $\chi$  via Weil representation. Denote by  $\pi_K$  the base change lifting of  $\sigma$  to  $\text{GL}_2(K)$ , then we have

$$L(s, \pi, \chi) = L(s, \pi_K \otimes \chi), \quad \epsilon(s, \pi, \chi) = \eta(-1)\epsilon(s, \pi_K \otimes \chi, \psi_K)$$

Note that  $\epsilon(\pi, \chi) := \epsilon(1/2, \pi, \chi)$  equals  $\pm 1$  and is independent of the choice of  $\psi$ . In the following, we denote  $L(s, \pi, \text{ad}) := L(s, \sigma, \text{ad})$  the adjoint  $L$ -factor of  $\sigma$ .

**3.1. Local Toric Integrals.** Let  $\mathcal{P}(\pi, \chi)$  denote the functional space

$$\mathcal{P}(\pi, \chi) := \text{Hom}_{K^\times}(\pi, \chi^{-1}).$$

By a theorem of Tunnell and Saito ([36], [27]), the space  $\mathcal{P}(\pi, \chi)$  has dimension at most one and equals one if and only if

$$\epsilon(\pi, \chi) = \chi\eta(-1)\epsilon(B).$$

**Lemma 3.1.** *Let the pair  $(\pi, \chi)$  be as above such that  $\epsilon(\pi, \chi) = \chi\eta(-1)\epsilon(B)$ .*

- (1) *If  $K$  is split or  $\pi$  is a principal series, then  $B$  is split.*
- (2) *Suppose  $K/F = \mathbb{C}/\mathbb{R}$ ,  $\sigma$  is the discrete series of weight  $k$ , and  $\chi(z) = |z|_{\mathbb{C}}^s(z/\sqrt{|z|_{\mathbb{C}}})^m$  with  $s \in \mathbb{C}$  and  $m \equiv k \pmod{2}$ . Then  $B$  is split if and only if  $m \geq k$ .*

Furthermore, assume  $F$  is nonarchimedean.

- (3) *If  $K/F$  is nonsplit and  $\sigma$  is the special representation  $\text{sp}(2) \otimes \mu$  with  $\mu$  a character of  $F^\times$ , then  $B$  is division if and only if  $\mu_K\chi = 1$  with  $\mu_K := \mu \circ N_{K/F}$ .*
- (4) *If  $K/F$  is inert and  $c = 0$ , then  $B$  is split if and only if  $n$  is even.*
- (5) *If  $K$  is nonsplit with  $c \geq n$ , then  $B$  is split.*

*Proof.* See Proposition 1.6, 1.7 in [36] for (1), (3) Proposition 6.5, 6.3 (2) in [14] for (2), (4). We now give a proof of (5). If  $\pi$  is a principal series, then by (1),  $B$  is split. If  $\sigma$  is a supercuspidal representation, then by [36] Lemma 3.1,  $B$  is split if  $n(\chi) \geq ne/2 + (2 - e)$ . It is then easy to check that if  $c \geq n$ , this condition always holds. Finally, assume  $\sigma = \text{sp}(2) \otimes \mu$  with  $\mu$  a character of  $F^\times$ . By (2),  $B$  is division if and only if  $\mu_K\chi = 1$ . If  $\mu$  is unramified, then  $n = 1$  and  $\chi$  is ramified which implies that  $B$  must be split. Assume  $\mu$  is ramified, then  $n = 2n(\mu)$  and by [36] Lemma 1.8,  $fn(\mu_K) = n(\mu) + n(\mu\eta) - n(\eta)$  where  $f$  is the residue degree of  $K/F$ . If  $K/F$  is unramified and  $\mu_K\chi = 1$ , then  $c = n(\mu_K) = n(\mu) = n/2$ , a contradiction. If  $K/F$  is ramified and  $\mu_K\chi = 1$ , then  $2c - 1 \leq n(\mu_K) < 2n(\mu) = n$ , a contradiction again. Hence, if  $c \geq n$ ,  $B$  is always split.  $\square$

Assume that the pair  $(\pi, \chi)$  is *essentially unitary* in the sense that there exists some character  $\mu = |\cdot|^s$  on  $F^\times$  with  $s \in \mathbb{C}$  such that both  $\pi \otimes \mu$  and  $\chi \otimes \mu_K^{-1}$  are unitary. In particular, if  $\pi$  is a local component of some global cuspidal representation, then  $(\pi, \chi)$  is essentially unitary. We shall only consider essentially unitary  $(\pi, \chi)$ . Under such an assumption, we study the space  $\mathcal{P}(\pi, \chi)$  via the following toric integral

$$\int_{F^\times \backslash K^\times} \langle \pi(t)f_1, f_2 \rangle \chi(t) dt$$

where  $f_1 \in \pi$ ,  $f_2 \in \tilde{\pi}$  and  $\langle \cdot, \cdot \rangle$  is any invariant pairing on  $\pi \times \tilde{\pi}$ . The following basic properties for this toric integral are established in [38]:

- It is absolutely convergent for any  $f_1 \in \pi$  and  $f_2 \in \tilde{\pi}$ ;
- $\mathcal{P}(\pi, \chi) \neq 0$  if and only if  $\mathcal{P}(\pi, \chi) \otimes \mathcal{P}(\tilde{\pi}, \chi^{-1}) \neq 0$  and in this case the above integral defines a generator of  $\mathcal{P}(\pi, \chi) \otimes \mathcal{P}(\tilde{\pi}, \chi^{-1})$ .
- for  $f_1 \in \pi, f_2 \in \tilde{\pi}$  such that  $\langle f_1, f_2 \rangle \neq 0$ , define the toric integral

$$\beta(f_1, f_2) := \frac{L(1, \eta)L(1, \pi, \text{ad})}{L(2, 1_F)L(1/2, \pi, \chi)} \int_{F^\times \backslash K^\times} \frac{\langle \pi(t)f_1, f_2 \rangle}{\langle f_1, f_2 \rangle} \chi(t) dt.$$

Then  $\beta(f_1, f_2) = 1$  in the case:  $B = M_2(F)$ ,  $K$  is an unramified extension over  $F$ , both  $\pi$  and  $\chi$  are unramified,  $dt$  is normalized such that  $\text{Vol}(\mathcal{O}_K^\times/\mathcal{O}^\times) = 1$ , and  $f_1, f_2$  are spherical.

Note that for any pair  $(\pi, \chi)$ , the above  $\beta$  integral is invariant if we modify  $(\pi, \chi)$  to  $(\pi \otimes \mu, \chi \otimes \mu_K^{-1})$  for any character  $\mu$  of  $F^\times$ . Therefore, we may assume  $\pi$  and  $\chi$  are both unitary from now on and identify  $(\tilde{\pi}, \chi^{-1})$  with  $(\tilde{\pi}, \bar{\chi})$ . Let  $(\cdot, \cdot) : \pi \times \pi \rightarrow \mathbb{C}$  be the Hermitian pairing defined by  $(f_1, f_2) = \langle f_1, \bar{f}_2 \rangle$ .

Denote  $\beta(f) := \beta(f, \bar{f})$ . Then the functional space  $\mathcal{P}(\pi, \chi)$  is nontrivial if and only if  $\beta$  is nontrivial. Assume  $\mathcal{P}(\pi, \chi)$  is nonzero in the following. A nonzero vector  $f$  of  $\pi$  is called a *test vector* for  $\mathcal{P}(\pi, \chi)$  if  $\ell(f) \neq 0$  for some (thus any) nonzero  $\ell \in \mathcal{P}(\pi, \chi)$ , or equivalently,  $\beta(f)$  is non-vanishing.

The notation of new vectors in an irreducible smooth admissible representation of  $\text{GL}_2(F)$  (see [7] for  $F$  non-archimedean and [23] for  $F$  archimedean) can be viewed as a special case of test vectors. Let  $\pi$  be an irreducible admissible representation of  $\text{GL}_2(F)$ . Recall the definition of *new vector line* in  $\pi$  as follows. Denote by  $T = K^\times$  the diagonal torus in  $\text{GL}_2(F)$ . Write  $T = ZT_1$  with  $T_1 = \left\{ \begin{pmatrix} * & \\ & 1 \end{pmatrix} \right\}$ .

- If  $F$  is nonarchimedean, then the new vector line is the invariant subspace  $\pi^{U_1(n)}$ .

- If  $F$  is archimedean, take  $U = \mathcal{O}_2(\mathbb{R})$  if  $F = \mathbb{R}$  and  $U_2$  if  $F = \mathbb{C}$ . The new vector line consists of vectors  $f \in \pi$  which are invariant under  $T_1 \cap U$  with weight minimal.

It is known that new vectors satisfy the following properties.

- (1) For any  $s \in \mathbb{C}$ , denote by  $\omega_s$  the character on  $T$  such that  $\omega_s|_Z = \omega$  and  $\omega_s|_{T_1} = |\cdot|^{s-1/2}$ . Then any nonzero  $f$  in the new vector line is a test vector for  $\mathcal{P}(\pi, \omega_s^{-1})$ ;
- (2) If denote by  $\mathcal{W}(\pi, \psi)$  the Whittaker model of  $\pi$  with respect to  $\psi$ , then there is a vector  $W_0$  in the new vector line, called the *normalized new vector* of  $\pi$  such that the local zeta integral  $|\delta|^{s-1/2} Z(s, W_0)$  equals  $L(s, \pi)$ .

### 3.2. Local Orders of Quaternions.

Assume  $F$  is nonarchimedean in this subsection.

First, in the case that the quaternion algebra  $B$  is split, given non-negative integers  $m$  and  $k$ , we want to classify all the  $K^\times$  conjugacy classes of Eichler orders  $R$  in  $B$  with discriminant  $m$  such that  $R \cap K = \mathcal{O}_k$ . For this, identify  $B$  with the  $F$ -algebra  $\text{End}_F(K)$  which contains  $K$  as an  $F$ -subalgebra by multiplication. Recall that an Eichler order in  $B$  is the intersection of two maximal orders in  $B$ . Then any Eichler order must be of form  $R(L_1, L_2) := R(L_1) \cap R(L_2)$  where  $L_i$ ,  $i = 1, 2$  are two  $\mathcal{O}$ -lattices in  $K$  and  $R(L_i) := \text{End}_{\mathcal{O}}(L_i)$ . Denote by  $d(L_1, L_2)$  its discriminant. For any maximal order  $R(L)$ , there exists a unique integer  $j \geq 0$  such that  $L = t\mathcal{O}_j$  for some  $t \in K^\times$ . In fact,  $\mathcal{O}_j = \{x \in K | xL \subset L\}$ . Thus any  $K^\times$ -conjugacy class of Eichler order contains an order of form  $R(\mathcal{O}_j, t\mathcal{O}_{j'})$  with  $0 \leq j' \leq j$  and  $t \in K^\times$  and the conjugacy class is exactly determined by the integers  $j' \leq j$  and the class of  $t \in K^\times$  modulo  $F^\times \mathcal{O}_{j'}^\times$ . The question is reduced to solving the equation with variables  $k'$  and  $[t]$ :

$$d(\mathcal{O}_k, t\mathcal{O}_{k'}) = m, \quad 0 \leq k' \leq k, \quad [t] \in K^\times / F^\times \mathcal{O}_{k'}^\times.$$

Note that if  $(k', [t])$  is a solution, then so is  $(k', [\bar{t}])$ . A complete representative system  $(k', t)$  with  $t \in K^\times$  for solutions to the above equation corresponds to a complete system  $R(\mathcal{O}_k, t\mathcal{O}_{k'})$  for  $K^\times$ -conjugacy classes of Eichler orders  $R$  with discriminant  $m$  and  $R \cap K = \mathcal{O}_k$ .

**Lemma 3.2.** *Let  $m, k$  be non-negative integers. Let  $\tau \in K^\times$  such that  $\mathcal{O}_K = \mathcal{O}[\tau]$ , if  $K$  split then  $\tau^2 - \tau = 0$ , and if  $K$  non-split then  $v(\tau) = (e - 1)/2$ . Denote by  $d := k + k' - m$ . Then a complete representative system of  $(k', t)$  is the following:*

- For  $0 \leq m \leq 2k$ ,  $k' \in [m - k, k]$  with  $d$  even, i.e.  $d \in 2 \cdot [0, k']$ , and

$$t = 1 + \varpi^{\frac{d}{2}} \tau u, \quad u \in (\mathcal{O} / \varpi^{k' - \frac{d}{2}} \mathcal{O})^\times.$$

*Note that in the case  $k' = k - m \geq 0$ , the unique class of  $t$  is also represented by 1.*

- For split  $K \cong F^2$  and  $k + 1 \leq m$ ,  $k' \in [0, \min(m - k - 1, k)]$ , i.e.  $d \in [k - m, 0)$ , and

$$t = (\varpi^{\pm d} u, 1), \quad u \in (\mathcal{O} / \varpi^{k'} \mathcal{O})^\times.$$

- For non-split  $K$  and  $k + 1 \leq m \leq 2k + e - 1$ ,  $k' = m - k - e + 1$ , i.e.  $d = 1 - e$ , and

$$t = \varpi x + \tau, \quad x \in \mathcal{O} / \varpi^{k' + e - 2} \mathcal{O}.$$

*Proof.* Note that the discriminant  $d(L_1, L_2)$  of the Eichler order  $R(L_1, L_2)$  can be computed as follows.

Let  $e_i, e'_i$  be an  $\mathcal{O}$ -basis of  $L_i$ ,  $i = 1, 2$ , and let  $A = (a_{ij}) \in \text{GL}_2(F)$  such that  $A \begin{pmatrix} e_1 \\ e'_1 \end{pmatrix} = \begin{pmatrix} e_2 \\ e'_2 \end{pmatrix}$ . Let  $v : F \rightarrow \mathbb{Z} \cup \{\infty\}$  be the additive valuation on  $F$  such that  $v(\varpi) = 1$ . Denote by  $\alpha = \min_{i,j} v(a_{ij})$  and  $\beta = v(\det A)$ . Then  $d(L_1, L_2) = |2\alpha - \beta|$ . Now solve the equation

$$d(\mathcal{O}_k, t\mathcal{O}_{k'}) = m, \quad k' \in [0, k], \quad t \in K^\times / F^\times \mathcal{O}_{k'}^\times.$$

□

Denote

$$c_1 = \begin{cases} 0, & \text{if } K \text{ is nonsplit and } c < n; \\ c, & \text{otherwise.} \end{cases}$$

**Lemma 3.3.** *There exists an order  $R$  of discriminant  $n$  and  $R \cap K = \mathcal{O}_{c_1}$  satisfying the condition: if  $nc_1 \neq 0$ , then  $R$  is the intersection of two maximal orders  $R'$  and  $R''$  of  $B$  such that  $R' \cap K = \mathcal{O}_{c_1}$ ,  $R'' \cap K = \mathcal{O}_{\max\{0, c_1 - n\}}$ . Such order is unique up to  $K^\times$ -conjugacy unless  $0 < c_1 < n$ . In the case  $0 < c_1 < n$ , there are exact two  $K^\times$ -conjugacy classes which are conjugate to each other by a normalizer of  $K^\times$ .*

*Proof.* If  $nc_1 = 0$ , this is proved in [14], Propositions 3.2 and 3.4. Now assume that  $nc_1 \neq 0$ , then  $B$  is split and one can apply Lemma 3.2. □

Let  $R$  be an  $\mathcal{O}$ -order of  $B$  of discriminant  $n$  such that  $R \cap K = \mathcal{O}_{c_1}$ . Such an order  $R$  is called admissible for  $(\pi, \chi)$  if the following conditions are satisfied

- (1) If  $nc_1 \neq 0$  (thus  $B$  is split), then  $R$  is the intersection of two maximal orders  $R'$  and  $R''$  of  $B$  such that  $R' \cap K = \mathcal{O}_{c_1}$  and  $R'' \cap K = \mathcal{O}_{\max\{0, c_1 - n\}}$ .
- (2) If  $0 < c_1 < n$ , fix an  $F$ -algebra isomorphism  $K \cong F^2$  and identify  $B$  with  $\text{End}_F(K)$ . Note that the two  $K^\times$ -conjugacy classes of  $\mathcal{O}$ -orders in  $B$  satisfying the above condition (1) contain respectively the orders  $R_i = R'_i \cap R''_i, i = 1, 2$  with  $R'_1 = R'_2 = \text{End}_{\mathcal{O}}(\mathcal{O}_c)$ ,  $R''_1 = \text{End}_{\mathcal{O}}((\varpi^{n-c}, 1)\mathcal{O}_K)$  and  $R''_2 = \text{End}_{\mathcal{O}}((1, \varpi^{n-c})\mathcal{O}_K)$ . Denote by  $\chi_1(a) = \chi(a, 1)$  and  $\chi_2(b) = \chi(1, b)$ . Then  $R$  is  $K^\times$ -conjugate to some  $R_i$  such that the conductor of  $\chi_i$  is  $c_1$ .

**Lemma 3.4.** *If  $K$  is nonsplit,  $n > 0$  and  $c = 0$ , then there is a unique admissible order  $R$  for  $(\pi, \chi)$ .*

*Proof.* Let  $\mathcal{O}_B$  be a maximal order containing  $\mathcal{O}_K$ , then by [14] (3.3), any admissible order for  $(\pi, \chi)$  is  $K^\times$ -conjugate to  $R := \mathcal{O}_K + I\mathcal{O}_B$  where  $I$  is a nonzero ideal of  $\mathcal{O}_K$  such that  $n = \delta(B) + \text{length}_{\mathcal{O}}(\mathcal{O}_K/I)$ . If  $B$  is nonsplit, then  $\mathcal{O}_B$  is invariant under  $B^\times$ -conjugations and  $R$  is unique. Assume  $B$  is split. As  $\mathcal{O}_K^\times \subset \mathcal{O}_B^\times$ ,  $\mathcal{O}_B$  is invariant under  $F^\times \mathcal{O}_K^\times$ -conjugations. In particular, if  $K$  is unramified, then  $K^\times = F^\times \mathcal{O}_K^\times$  and  $R$  is unique. Consider the case  $K$  is ramified. Then  $K^\times = F^\times \mathcal{O}_K^\times \cup \varpi_K F^\times \mathcal{O}_K^\times$  and it reduces to show that  $\varpi_K$  normalizes  $R$ . For this, embed  $K$  into  $B = M_2(F)$  by  $\varpi_K \mapsto \begin{pmatrix} \text{tr} \varpi_K & 1 \\ -N \varpi_K & 0 \end{pmatrix}$  and take  $\mathcal{O}_B = M_2(\mathcal{O})$ . Then  $R = \mathcal{O}_K + \varpi_K^n M_2(\mathcal{O})$ . Note that  $R_0(1) = \mathcal{O}_K + \varpi_K M_2(\mathcal{O})$  with  $R_0(1) = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} \end{pmatrix}$  the Iwahori order in  $M_2(F)$ . Denote  $m$  the maximal integer such that  $2m \leq n$ . Then  $R = \mathcal{O}_K + \varpi_K^{m-1} \varpi_K R_0(1)$  (resp.  $R = \mathcal{O}_K + \varpi_K^m R_0(1)$ ) if  $n$  is even (resp.  $n$  is odd). As  $\varpi_K$  normalizes  $R_0(1)$ , it also normalizes  $R$  and  $R$  is unique.  $\square$

In the following, take an admissible  $\mathcal{O}$ -order  $R$  of  $B$ . Let  $U = R^\times$  and define

$$\gamma := \frac{\text{Vol}(U)}{\text{Vol}(U_0(n))},$$

where the Haar measure is given so that  $\text{Vol}(\text{GL}_2(\mathcal{O})) = L(2, 1_F)^{-1} |\delta|^2$  and  $\text{Vol}(\mathcal{O}_B^\times) = L(2, 1_F)^{-1} (q - 1)^{-1} |\delta|^2$  if  $B$  is division.

**Lemma 3.5.** *If either  $R$  is not maximal or  $B$  is nonsplit, then*

$$\gamma = L(1, 1_F)(1 - e(R)q^{-1})$$

where  $e(R)$  is the Eichler symbol of  $R$ , which is defined as follows. Let  $\kappa(R) = R/\text{rad}(R)$  with  $\text{rad}(R)$  the Jacobson radical of  $R$  and let  $\kappa$  be the residue field of  $F$ . Then

$$e(R) = \begin{cases} 1, & \text{if } \kappa(R) = \kappa^2, \\ -1, & \text{if } \kappa(R) \text{ is a quadratic field extension of } \kappa, \\ 0, & \text{if } \kappa(R) = \kappa. \end{cases}$$

*Proof.* Let  $R_0$  be a maximal order of  $B$  containing  $R$ . Then we have the following formula (for example, see [41]):

$$\frac{[R_0^\times : R^\times]}{[R_0 : R]} = \frac{|\kappa(R_0)^\times|/|\kappa(R^\times)|}{|\kappa(R_0)|/|\kappa(R)|}.$$

If  $B$  is split and  $R$  is not maximal, then

$$[R_0 : R] = q^n, \quad \frac{|\kappa(R_0)^\times|}{|\kappa(R_0)|} = (1 - q^{-2})(1 - q^{-1}), \quad \frac{|\kappa(R)|}{|\kappa(R)^\times|} = (1 - q^{-1})^{-1}(1 - e(R)q)^{-1},$$

while if  $B$  is division, then

$$[R_0 : R] = q^{n-1}, \quad \frac{|\kappa(R_0)^\times|}{|\kappa(R_0)|} = 1 - q^{-2}, \quad \frac{|\kappa(R)|}{|\kappa(R)^\times|} = (1 - q^{-1})^{-1}(1 - e(R)q)^{-1}.$$

Summing up,

$$[R_0^\times : R^\times] = (q - 1)^{-\delta(B)} q^n (1 - q^{-2})(1 - e(R)q^{-1})^{-1},$$



where  $\delta(B) = 0$  (resp. 1) if  $B$  is split (resp. ramified). Thus

$$\begin{aligned}\gamma^{-1} &= \frac{\text{Vol}(U_0(n))}{\text{Vol}(U)} = \frac{\text{Vol}(\text{GL}_2(\mathcal{O}))}{\text{Vol}(R_0^\times)} \frac{[R_0^\times : U]}{[\text{GL}_2(\mathcal{O}) : U_0(n)]} \\ &= \frac{L(2, 1)^{-1}}{(q-1)^{-\delta(B)} L(2, 1)^{-1}} \frac{(q-1)^{-\delta(B)} q^n (1-q^{-2})(1-e(R)q^{-1})^{-1}}{q^n (1-q^{-2})(1-q^{-1})^{-1}} \\ &= L(1, 1_F)^{-1} (1-e(R)q^{-1})^{-1}.\end{aligned}$$

□

### 3.3. Test Vector Spaces.

**Definition 3.6.** Define  $V(\pi, \chi) \subset \pi$  to be the subspace of vectors  $f$  satisfying the following condition:

- for nonarchimedean  $F$ ,  $K$  split or  $c \geq n$ , let  $U \subset G$  be the compact subgroup defined before Lemma 3.5, then  $f$  is  $\omega$ -eigen under  $U$ . Here, write  $U = (U \cap Z)U'$  such that  $U' = U$  if  $cn = 0$  and  $U' \cong U_1(n)$  otherwise, and view  $\omega$  as a character on  $U \cap Z$  and extends to  $U$  by making it trivial on  $U'$ ;
- for nonarchimedean  $F$ ,  $K$  nonsplit and  $c < n$ ,  $f$  is  $\chi^{-1}$ -eigen under the action of  $K^\times$ ;
- for archimedean  $F$ , let  $U$  be a maximal compact subgroup of  $G$  such that  $U \cap K^\times$  is the maximal compact subgroup of  $K^\times$ , then  $f$  is  $\chi^{-1}$ -eigen under  $U \cap K^\times$  with weight minimal.

**Proposition 3.7.** The dimension of  $V(\pi, \chi)$  is one and any nonzero vector in  $V(\pi, \chi)$  is a test vector for  $\mathcal{P}(\pi, \chi)$ .

*Proof.* If  $F$  is nonarchimedean, the claim that  $\dim V(\pi, \chi) = 1$  follows from local new form theory [7]. Assume  $F$  is archimedean. If  $K$  is nonsplit, then  $V(\pi, \chi)$  is the  $\chi^{-1}$ -eigen line. If  $K$  is split, without loss of generality, embed  $K^\times$  into  $G \cong \text{GL}_2(F)$  as the diagonal matrices and decompose  $K^\times = F^\times K^1$  such that the image of  $K^1$  in  $G$  is  $\left\{ \begin{pmatrix} * & \\ & 1 \end{pmatrix} \right\}$ . Then  $V(\pi, \chi)$  is the new vector line for  $\pi \otimes \chi_1$  with  $\chi_1 := \chi|_{K^1}$ .

We shall prove any nonzero vector in  $V(\pi, \chi)$  is a test vector in next subsection by computing the toric integral  $\beta$ . □

**Proposition 3.8.** Assume  $K/F$  is a quadratic extension of nonarchimedean fields with  $n > 0$  and  $c = 0$ . Then  $V(\pi, \chi) \subseteq \pi^{R^\times}$  and  $\dim \pi^{R^\times} = \dim \pi^{\mathcal{O}_K^\times} \leq 2$ . The dimension of  $\pi^{R^\times}$  is one precisely when  $K/F$  is inert or  $K/F$  is ramified and  $\epsilon(\pi, \chi_1) \neq \epsilon(\pi, \chi_2)$  where  $\chi_i, i = 1, 2$ , are unramified characters of  $K^\times$  with  $\chi_i|_{F^\times} \cdot \omega = 1$ .

The proof of this proposition is referred to [14] and [15] except for the case that  $\pi$  is a supercuspidal representation on  $G = \text{GL}_2(F)$ . For this case, the proof in [14], §7 is based on a character formula for odd residue characteristic. We next prove this case with arbitrary residue characteristic.

Let  $R_0 = M_2(\mathcal{O})$  if  $e = 1$  and the Iwahori order  $\begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} \end{pmatrix}$  if  $e = 2$ . Fix an embedding of  $K$  into  $M_2(F)$  such that  $R_0 \cap K = \mathcal{O}_K$ . Consider the following filtration of open compact subgroups of  $G$  and  $K^\times$ :

$$\mathcal{K}(r) := (1 + \varpi^r R_0) \cap \text{GL}_2(\mathcal{O}), \quad \mathcal{E}(r) := \mathcal{K}(r) \cap K^\times, \quad r \geq 0.$$

Denote  $m$  the minimal integer such that  $2m + 1 \geq n$ . The proof is based on the following proposition.

**Proposition 3.9.** For any integer  $r \geq m$ ,  $\pi^{\mathcal{K}(r)} = \pi^{\mathcal{E}(r)}$ .

*Proof.* Firstly, note that it is enough to prove Proposition 3.9 for the case  $\pi$  is minimal, that is,  $\pi$  has minimal conductor among its twists. In fact, assume  $\pi$  is not minimal. Denote  $n_0$  the minimal conductor of  $\pi$ . Take a character  $\mu$  so that  $\pi_0 := \pi \otimes \mu$  has conductor  $n_0$ . Then by [35], Proposition 3.4,  $n_0 \leq \max(n, 2n(\mu))$  with equality if  $\pi$  is minimal or  $n \neq 2n(\mu)$ . In particular,  $n = 2m$  with  $n(\mu) = m$ . Hence, for any  $r \geq m$ ,  $\pi^{\mathcal{K}(r)} = \pi_0^{\mathcal{K}(r)}$ ,  $\pi^{\mathcal{E}(r)} = \pi_0^{\mathcal{E}(r)}$ . Note that  $r \geq n_0/2$  and one then can apply the proposition for the minimal representation  $\pi_0$ .

Assume  $\pi$  is minimal in the following. As  $\mathcal{K}(r) \supset \mathcal{E}(r)$ ,  $\pi^{\mathcal{K}(r)} \subset \pi^{\mathcal{E}(r)}$ . It remains to prove that  $\pi^{\mathcal{K}(r)}$  and  $\pi^{\mathcal{E}(r)}$  have the same dimension. Denote  $\pi_D$  the representation on  $D^\times$  where  $D$  is the division quaternion algebra over  $F$  so that the Jacquet-Langlands lifting of  $\pi_D$  to  $G$  is  $\pi$ . Then  $\pi_D$  has conductor  $n$ , that is,  $\pi_D^{1+\varpi_D^{n-1}\mathcal{O}_D} = \pi_D$  and  $\pi_D^{1+\varpi_D^{n-2}\mathcal{O}_D} = 0$  where  $\varpi_D$  is a uniformizer of  $D$ . Moreover, by [6],

Proposition 6.5,

$$\dim \pi_D = \begin{cases} 2q^{m-1}; & \text{if } n \text{ is even;} \\ q^m + q^{m-1}; & \text{if } n \text{ is odd.} \end{cases}$$

Note that for any  $r \geq m$ ,  $\mathcal{E}(r) \subset (1 + \varpi_D^{n-1} \mathcal{O}_D) \cap \mathcal{O}_K^\times$ . Therefore, by Tunnell-Saito's theorem, if we denote  $\mathcal{X}(r)$  the set of all the characters  $\mu$  on  $K^\times$  such that  $\mu|_{F^\times} \omega = 1$  and  $\mu|_{\mathcal{E}(r)} = 1$ , then

$$\dim \pi^{\mathcal{E}(r)} + \dim \pi_D = \sum_{\mu \in \mathcal{X}(r)} \dim \pi^\mu + \sum_{\mu} \dim \pi_D^\mu = \sum_{\mu \in \mathcal{X}(r)} (\dim \pi^\mu + \dim \pi_D^\mu) = \#\mathcal{X}(r)$$

and on the other hand, the lemma below implies that

$$\dim \pi^{\mathcal{K}(r)} + \dim \pi_D = \#\mathcal{X}(r),$$

and then the equality  $\dim \pi^{\mathcal{E}(r)} = \dim \pi^{\mathcal{K}(r)}$  holds.  $\square$

**Lemma 3.10.** *Let  $\pi$  be minimal. For any integer  $r \geq m$ , we have the following dimension formula*

$$\dim \pi^{\mathcal{K}(r)} = \begin{cases} q^r + q^{r-1} - 2q^{m-1}; & \text{if } n \text{ is even and } e = 1; \\ q^r + q^{r-1} - (q^{m-1} + q^{m-2}); & \text{if } n \text{ is odd and } e = 1; \\ 2q^r - (q^m + q^{m-1}); & \text{if } n \text{ is odd and } e = 2; \\ 2q^r - 2q^{m-1}; & \text{if } n \text{ is even and } e = 2. \end{cases}$$

*Proof.* For  $r = m$  and  $e = 1$ , this formula occurs in [8], Theorem 3. We now use the method in [8] to prove the dimension formula for the case  $n$  is even and  $e = 1$  while other cases are similar. Firstly, recall some basics about Kirillov model. Let  $\psi$  be an unramified additive character of  $F$ . Associated to  $\psi$ , we can realize  $\pi$  on the space  $C_c^\infty(F^\times)$  of Schwartz functions on the multiplicative group. For any  $f \in C_c^\infty(F^\times)$  and any character  $\mu$  of  $\mathcal{O}^\times$ , define

$$f_k(\mu) = \int_{\mathcal{O}^\times} f(u\varpi^k) \mu(u) du$$

where we choose the Haar measure on  $\mathcal{O}^\times$  such that the total measure is 1. Define further the formal power series

$$\hat{f}(\mu, t) = \sum_{k \in \mathbb{Z}} f_k(\mu) t^k$$

which is actually a Laurent polynomial in  $t$  as  $f$  has compact support on  $F^\times$ . Because  $f$  is locally constant, this vanishes identically for all but a finite number of  $\mu$ . By Fourier duality for  $F^\times$ , knowing  $\hat{f}(\mu, t)$  for all  $\mu$  is equivalent to knowing  $f$ . For each  $\mu$ , there is a formal power series  $C(\mu, t)$  such that for all  $f \in C_c^\infty(F^\times)$

$$(\pi(w)f)^\wedge(\mu, t) = C(\mu, t) \hat{f}(\mu^{-1} \omega_0^{-1}, t^{-1} z_0^{-1}), \quad C(\mu, t) = C_0(\mu) t^{n_\mu}, \quad w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

where  $\omega_0 = \omega|_{\mathcal{O}^\times}$ ,  $z_0 = \omega(\varpi)$  and some integer  $n_\mu \leq -2$ . Moreover, if  $\mu = 1$ , then  $-n_1 = n$ . For any character  $\mu$  of  $\mathcal{O}^\times$ ,

$$-n_\mu = \begin{cases} n, & \text{if } n(\mu) \leq m; \\ 2n(\mu), & \text{if } n(\mu) > m. \end{cases}$$

In fact, if take any character  $\Omega$  on  $F^\times$  so that  $\Omega|_{\mathcal{O}^\times} = \mu$ , denote  $\pi' = \pi \otimes \Omega$  and  $C'(\cdot, \cdot)$  the monomial occurred in the above functional equation, then for any character  $\nu$  on  $\mathcal{O}^\times$ ,  $C'(\nu, t) = C(\nu\mu, \Omega(\varpi)t)$ . Therefore,  $-n_\mu = n(\pi') = \max(n, 2n(\mu))$ .

On the other hand, by [8], Corollary to Lemma 2, for any  $r \geq m$ , the subspace  $\pi^{\mathcal{K}(r)}$  is isomorphic to the space of all functions  $f(\mu, t)$  such that

- (1)  $\hat{f}(\mu, t) = 0$  unless  $n(\mu) \leq r$ ;
- (2) for each  $\mu$ ,  $f_k(\mu) = 0$  unless  $-r \leq k \leq n_\mu + r$ .

Summing up, for a given  $\mu$  with conductor  $n(\mu) \leq r$ , the dimension of the space consisting of  $\hat{f}(\mu, t)$  with  $f \in \pi^{\mathcal{K}(r)}$  is

$$\begin{cases} 2(r - m) + 1; & \text{if } n(\mu) \leq m; \\ 2(r - n(\mu)) + 1; & \text{if } n(\mu) > m. \end{cases}$$

Therefore,

$$\begin{aligned}\dim \pi^{\mathcal{K}(r)} &= (q^m - q^{m-1})(2(r-m) + 1) + \sum_{m < k \leq r} (q^k - 2q^{k-1} + q^{k-2})(2(r-k) + 1) \\ &= q^r + q^{r-1} - 2q^{m-1}.\end{aligned}$$

□

*Proof of Proposition 3.8.* Note that  $R^\times = \mathcal{O}_K^\times \mathcal{K}(m)$  unless  $K$  is ramified with  $n$  even and once this equation holds, Proposition 3.8 follows directly from Proposition 3.9. It remains to consider the case  $K$  is ramified with  $n$  even. For this case,  $R^\times = \mathcal{O}_K^\times \mathcal{K}'(m)$  with  $\mathcal{K}'(m) = 1 + \varpi_K^{2m-1} R_0$ . We want to show  $\pi^{\mathcal{K}'(m)} = \pi^{\mathcal{E}'(m)}$  with  $\mathcal{E}'(m) = \mathcal{K}'(m) \cap K^\times$  and Proposition 3.8 then holds. By [36], Proposition 3.5,  $\pi$  is not minimal. Take a character  $\mu$  so that  $\pi_0 = \pi \otimes \mu$  has minimal conductor  $n_0$ . Then  $n(\mu) = m$ . Apply Proposition 3.9,

$$\pi^{\mathcal{K}'(m)} = \pi_0^{\mathcal{K}'(m)} \supset \pi_0^{\mathcal{K}(m-1)} = \pi_0^{\mathcal{E}(m-1)}.$$

We claim that  $\pi_0^{\mathcal{E}(m-1)} = \pi_0^{\mathcal{E}'(m)}$ . If so,  $\pi_0^{\mathcal{E}(m-1)} = \pi_0^{\mathcal{E}'(m)}$  and then  $\pi^{\mathcal{K}'(m)} = \pi^{\mathcal{E}'(m)}$ . To prove this, note that  $\mathcal{E}'(m) \subset \mathcal{E}(m-1) \subset 1 + \varpi_D^{n_0-1} \mathcal{O}_D$ . Use Tunnell-Saito's theorem,

$$\dim \pi_0^{\mathcal{E}(m-1)} + \dim \pi_{0,D} = \#\mathcal{X}(m-1), \quad \dim \pi_0^{\mathcal{E}'(m)} + \dim \pi_{0,D} = \#\mathcal{X}'(m)$$

where the set  $\mathcal{X}(m-1)$  consists of characters  $\Omega$  of  $K^\times$  such that  $\Omega|_{F^\times} \cdot \omega_{\pi_0} = 1$  with  $\Omega|_{\mathcal{E}(m-1)} = 1$  and the set  $\mathcal{X}'(m)$  is defined similarly. As they are nonempty,

$$\#\mathcal{X}(m-1) = \#K^\times / F^\times \mathcal{E}(m-1) = \#K^\times / F^\times \mathcal{E}'(m) = \#\mathcal{X}'(m).$$

Thus  $\pi_0^{\mathcal{E}(m-1)} = \pi_0^{\mathcal{E}'(m)}$  and the proof is complete. □

**3.4. Local Computations.** Let  $\mathcal{W}(\sigma, \psi)$  be the Whittaker model of  $\sigma$  with respect to  $\psi$  and recall that we have an invariant Hermitian form on  $\mathcal{W}(\sigma, \psi)$  defined by

$$(W_1, W_2) := \int_{F^\times} W_1 \left[ \begin{pmatrix} a & \\ & 1 \end{pmatrix} \right] \overline{W_2 \left[ \begin{pmatrix} a & \\ & 1 \end{pmatrix} \right]} d^\times a.$$

For any  $W \in \sigma$ , denote

$$\alpha(W) = \frac{(W, W)}{L(1, \sigma, \text{ad}) L(1, 1_F) L(2, 1_F)^{-1}}.$$

**Proposition 3.11.** Denote  $W_0$  the normalized new vector of  $\sigma$ . If  $F$  is nonarchimedean, then

$$\alpha(W_0) |\delta|^{1/2} = \begin{cases} 1, & \text{if } \sigma \text{ is unramified;} \\ L(2, 1_F) L(1, 1_F)^{-1} L(1, \sigma, \text{ad})^{-\delta_\sigma}, & \text{otherwise,} \end{cases}$$

where  $\delta_\sigma \in \{0, 1\}$  and equals 0 precisely when  $\sigma$  is a subrepresentation of induced representation  $\text{Ind}(\mu_1, \mu_2)$  with at least one  $\mu_i$  unramified. If  $F = \mathbb{R}$  and  $\sigma$  is the discrete series  $\mathcal{D}_\mu(k)$ , then  $\alpha(W_0) = 2^{-k}$ .

The above proposition follows from the explicit form of  $W_0$ . If  $F$  is nonarchimedean,  $W_0$  is the one in the new vector line such that

$$W_0 \left[ \begin{pmatrix} \delta^{-1} & \\ & 1 \end{pmatrix} \right] = |\delta|^{-1/2}$$

and we have the following list (See [26], p.23)

(1) If  $\sigma = \pi(\mu_1, \mu_2)$  is a principal series, then

$$W_0 \left[ \begin{pmatrix} y & \\ & 1 \end{pmatrix} \right] = \begin{cases} |y|^{1/2} \sum_{k+l=v(y\delta)} \mu_1(\varpi)^k \mu_2(\varpi)^l 1_{\mathcal{O}}(\delta y), & \text{if } n(\mu_1) = n(\mu_2) = 0; \\ |y|^{1/2} \mu_1(\delta y) 1_{\mathcal{O}}(\delta y), & \text{if } n(\mu_1) = 0 \text{ and } n(\mu_2) > 0; \\ |\delta|^{-1/2} 1_{\mathcal{O}^\times}(\delta y), & \text{if } n(\mu_1) > 0 \text{ and } n(\mu_2) > 0. \end{cases}$$

(2) If  $\sigma = \text{sp}(2) \otimes \mu$  is a special representation, then

$$W_0 \left[ \begin{pmatrix} y & \\ & 1 \end{pmatrix} \right] = \begin{cases} |\delta|^{-1/2} \mu(\delta y) |\delta y| 1_{\mathcal{O}}(\delta y), & \text{if } n(\mu) = 0; \\ |\delta|^{-1/2} 1_{\mathcal{O}^\times}(\delta y), & \text{if } n(\mu) > 0. \end{cases}$$

(3) If  $\sigma$  is supercuspidal, then

$$W_0 \left[ \begin{pmatrix} y & \\ & 1 \end{pmatrix} \right] = |\delta|^{-1/2} 1_{\mathcal{O}^\times}(\delta y).$$

If  $F = \mathbb{R}$  and  $\sigma$  is the discrete series  $\mathcal{D}_\mu(k)$ , then

$$W_0 \left[ \begin{pmatrix} y & \\ & 1 \end{pmatrix} \right] = |y|^{k/2} e^{-2\pi|y|},$$

and in general, for archimedean cases, it is expressed by the Bessel function [23]. Note that for  $F = \mathbb{R}$  and  $\sigma$  a unitary discrete series of weight  $k$ , let  $W \in \mathcal{W}(\sigma, \psi)$  be the vector satisfying

$$W \left[ \begin{pmatrix} y & \\ & 1 \end{pmatrix} \right] = |y|^{k/2} e^{-2\pi|y|} 1_{\mathbb{R}_+^\times}(y).$$

Then  $W$  can be realized as a local component of a Hilbert newform and

$$(W_0, W_0) = 2(W, W), \quad Z(s, W) = \frac{1}{2} L(s, \sigma).$$

**Proposition 3.12.** *If  $F$  is nonarchimedean, let  $f$  be a nonzero vector in the one-dimensional space  $V(\pi, \chi)$ , then*

$$\beta(f)|D\delta|^{-1/2} = \begin{cases} 1, & \text{if } n = c = 0; \\ L(1, \eta)^2 |\varpi^c|, & \text{if } n = 0 \text{ and } c > 0; \\ \frac{L(1, 1_F)}{L(2, 1_F)} L(1, \pi, \text{ad})^{\delta_\pi}, & \text{if } n > 0, c = 0 \text{ and } K \text{ is split}; \\ \frac{L(1, 1_F)}{L(2, 1_F)} L(1, \eta)^2 |\varpi^c| \frac{L(1, \pi, \text{ad})^{\delta_\pi}}{L(1/2, \pi, \chi)}, & \text{if } nc > 0, \text{ either } K \text{ is split or } c \geq n; \\ e(1 - q^{-e}) \frac{L(1, \pi, \text{ad})}{L(1/2, \pi, \chi)}, & \text{if } n > c \text{ and } K \text{ is nonsplit.} \end{cases}$$

which is independent of the choice of  $f \in V(\pi, \chi)$ .

The proof of Proposition 3.12 is reduced to computing the integral

$$\beta^0 = \int_{F^\times \backslash K^\times} \frac{(\pi(t)f, f)}{(f, f)} \chi(t) dt,$$

where  $f$  is any nonzero vector in  $V(\pi, \chi)$ .

In the case  $n > c$  and  $K$  is nonsplit,  $f$  is  $\chi^{-1}$ -eigen and it is easy to see that  $\beta^0 = \text{Vol}(F^\times \backslash K^\times)$ .

From now on assume  $n \leq c$  or  $K$  is split. Then  $B = M_2(F)$  by Lemma 3.1 (5). Recall that the space  $V(\pi, \chi)$  depends on a choice of an admissible order  $R$  for  $(\pi, \chi)$ . Let  $f$  be a test vector in  $V(\pi, \chi)$  defined by  $R$ . For any  $t \in K^\times$ ,  $f' := \pi(t)f$  is a test vector defined by the admissible order  $R' = tRt^{-1}$ . It is easy to check that  $\beta(f') = \beta(f)$ . Thus, for a  $K^\times$ -conjugacy class of admissible orders, we can pick a particular order to compute  $\beta^0$ . Note that there is a unique  $K^\times$ -conjugacy class of admissible orders unless in the exceptional case  $0 < c_1 < n$  and  $n(\chi_1) = n(\chi_2) = c$ . In the exceptional case, there are exactly two  $K^\times$ -conjugacy classes of admissible orders, which are conjugate to each other by a normalizer of  $K^\times$  in  $B^\times$ .

Any admissible order (in the case  $n \leq c$  or  $K$  split) is an Eichler order of discriminant  $n$ , i.e. conjugate to  $R_0(n) := \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{p}^n & \mathcal{O} \end{pmatrix}$ . Choose an embedding of  $K$  into  $M_2(F)$  as follows such that  $R_0(n)$  is an admissible order for  $(\pi, \chi)$ .

- (1) If  $K$  is split, fix an  $F$ -algebra isomorphism  $K \cong F^2$ . If  $c \geq n$  or  $n(\chi_1) = c$ , embed  $K$  into  $M_2(F)$  by

$$\iota_1 : (a, b) \mapsto \gamma_c^{-1} \begin{pmatrix} a & \\ & b \end{pmatrix} \gamma_c, \quad \gamma_c = \begin{pmatrix} 1 & \varpi^{-c} \\ & 1 \end{pmatrix}.$$

If  $n(\chi_1) < c < n$ , embed  $K$  into  $M_2(F)$  by

$$\iota_2 : (a, b) \mapsto \gamma_c^{-1} \begin{pmatrix} b & \\ & a \end{pmatrix} \gamma_c.$$

Note that for any  $t \in K^\times$ ,  $\iota_1(t) = j\iota_2(t)j^{-1}$  with  $j = \gamma_c^{-1}w\gamma_c$  and  $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ .

- (2) If  $K$  is a field, take  $\tau \in \mathcal{O}_K$  such that  $\mathcal{O}_K = \mathcal{O}[\tau]$  and that if  $K/F$  is ramified then  $\tau$  is a uniformizer. Embed  $K$  into  $M_2(F)$  by

$$a + b\tau \mapsto \gamma_c^{-1} \begin{pmatrix} a + b\text{tr}\tau & bN\tau \\ -b & a \end{pmatrix} \gamma_c, \quad \gamma_c = \begin{pmatrix} \varpi^c N\tau & \\ & 1 \end{pmatrix}.$$

Assume  $K \cong F^2$ . Note that if  $n(\chi_1) < c < n$ ,

$$\beta^0 = \int_{F^\times \setminus K^\times} \frac{(\pi(\iota_2(t))W_0, W_0)}{(W_0, W_0)} \chi(t) dt = \int_{F^\times \setminus K^\times} \frac{(\pi(\iota_1(t))W_0, W_0)}{(W_0, W_0)} \bar{\chi}(t) dt$$

where  $\bar{\chi}_1 = \chi_2$ ,  $\bar{\chi}_2 = \chi_1$  and  $n(\bar{\chi}_1) = n(\chi_2) = c$ . We reduce to consider the case  $c \geq n$  or  $n(\chi_1) = c$ . Note that for the exceptional case, if we take  $\pi(j)W_0$  as a test vector, then

$$\beta^0 = \int_{F^\times \setminus K^\times} \frac{(\pi(\iota_1(t)j)W_0, \pi(j)W_0)}{(W_0, W_0)} \chi(t) dt = \int_{F^\times \setminus K^\times} \frac{(\pi(\iota_1(t))W_0, W_0)}{(W_0, W_0)} \chi(\bar{t}) dt$$

with  $n(\bar{\chi}_1) = n(\chi_2) = c$ . Thus, even for the exceptional case, only need to consider  $W_0$  as a test vector. Thus

$$\begin{aligned} \beta^0 &= (W_0, W_0)^{-1} \iint_{(F^\times)^2} \pi(\gamma_c)W_0 \left[ \begin{pmatrix} ab & \\ & 1 \end{pmatrix} \right] \overline{\pi(\gamma_c)W_0 \left[ \begin{pmatrix} b & \\ & 1 \end{pmatrix} \right]} \chi_1(a) d^\times b d^\times a \\ &= (W_0, W_0)^{-1} |Z(1/2, \pi(\gamma_c)W_0, \chi_1)|^2. \end{aligned}$$

If  $c = 0$ ,  $Z(1/2, W_0, \chi_1) = \chi_1(\delta)^{-1} L(1/2, \pi \otimes \chi_1)$  and then  $\beta^0 = (W_0, W_0)^{-1} L(1/2, \pi, \chi)$ . If  $c > 0$ , then

$$\begin{aligned} Z(1/2, \pi(\gamma_c)W_0, \chi_1) &= \int_{F^\times} W_0 \left[ \begin{pmatrix} a & \\ & 1 \end{pmatrix} \right] \psi(a\varpi^{-c}) \chi_1(a) d^\times a \\ &= \sum_{k \in \mathbb{Z}} W_0 \left[ \begin{pmatrix} \varpi^k & \\ & 1 \end{pmatrix} \right] \int_{\varpi^k \mathcal{O}^\times} \psi(a\varpi^{-c}) \chi_1(a) d^\times a. \end{aligned}$$

Assume  $n(\chi_1) = c$ , then the integral  $\int_{\varpi^k \mathcal{O}^\times} \psi(a\varpi^{-c}) \chi_1(a) d^\times a$  vanishes unless  $k = -v(\delta)$  while

$$\left| \int_{\delta^{-1} \mathcal{O}^\times} \psi(a\varpi^{-c}) \chi_1(a) d^\times a \right| = L(1, 1_F) |\delta|^{1/2} q^{-c/2}.$$

Thus

$$\beta^0 = (W_0, W_0)^{-1} L(1, 1_F)^2 q^{-c}.$$

Assume  $c \geq n$  and  $n(\chi_1) < c$ . Let  $j$  be a normalizer of  $K^\times$  with  $jt = \bar{t}j$  for any  $t \in K^\times$ . As  $c \geq n$ , there exists some  $t_0 \in K^\times$  such that  $t_0 U_0(n) t_0^{-1} = j U_0(n) j^{-1}$  and  $\pi(t_0)W_0, \pi(j)W_0$  are in the same line. Thus

$$\beta^0 = \int_{F^\times \setminus K^\times} \frac{(\pi(t)W_0, W_0)}{(W_0, W_0)} \bar{\chi}(t) dt = (W_0, W_0)^{-1} L(1, 1_F)^2 q^{-c}$$

as  $n(\bar{\chi}_1) = n(\chi_2) = c$ .

**Remark.** Assume  $n(\chi_1) < c < n$  and  $R$  is the intersection of two maximal orders  $R'$  and  $R''$  with  $R' \cap K = \mathcal{O}_c$  and  $R' \cap K = \mathcal{O}_K$ . If  $R$  is not admissible, then the toric integral for  $f$  is  $\omega$ -eigen under  $R^\times$  must vanish if  $c > 1$ . In the case  $c = 1$  and then  $n(\chi_1) = 0$ ,

$$\int_{F^\times \setminus K^\times} \frac{(\pi(\iota_1(t))W_0, W_0)}{(W_0, W_0)} \chi(t) dt = (W_0, W_0)^{-1} L(1, 1_F)^2 q^{-2}.$$

It remains to consider the case  $K$  is a field and  $c \geq n$ . Let  $\Psi(g)$  denote the matrix coefficient:

$$\Psi(g) := \frac{(\pi(g)W_0, W_0)}{(W_0, W_0)}, \quad g \in \mathrm{GL}_2(F).$$

Then

$$\beta^0 = \frac{\mathrm{Vol}(K^\times/F^\times)}{\#K^\times/F^\times \mathcal{O}_c^\times} \sum_{t \in K^\times/F^\times \mathcal{O}_c^\times} \Psi(t) \chi(t).$$

In this case  $c = 0$ ,  $\pi$  is unramified. Further, if  $K/F$  is unramified, then  $\beta^0 = \mathrm{Vol}(K^\times/F^\times) = |\delta|^{1/2}$ , and if  $K/F$  is ramified,  $\beta^0 = |D\delta|^{1/2} (1 + \Psi(\tau) \chi(\tau))$ , where  $\Psi(\tau)$  is expressed by the MacDonald polynomial and one has  $\beta(f) = |D\delta|^{1/2}$ . It remains to consider the case  $c > 0$ . Denote

$$S_i = \{1 + b\tau, b \in \mathcal{O}/\mathfrak{p}^c, v(b) = i\}, \quad 0 \leq i \leq c-1$$

and

$$S' = \begin{cases} \{a + \tau, a \in \mathfrak{p}/\mathfrak{p}^c\}, & \text{if } e = 1; \\ \{a\varpi + \tau, a \in \mathcal{O}/\mathfrak{p}^c\}, & \text{if } e = 2. \end{cases}$$

Then a complete representatives of  $K^\times/F^\times \mathcal{O}_c^\times$  can be taken as

$$\{1\} \sqcup (\sqcup_i S_i) \sqcup S'.$$



Note that  $\Psi$  is a function on  $U_1(n) \backslash G/U_1(n)$ . The following observation is key to our computation: the images of  $S_i$ ,  $0 \leq i \leq c-1$  and  $S'$  under the natural map

$$\text{pr} : K^\times \rightarrow U_1(n) \backslash G/U_1(n)$$

are constant. Precisely,

$$\text{pr}(S_i) = \left[ \begin{pmatrix} 1 & \varpi^{i-c} \\ & 1 \end{pmatrix} \right], \quad \text{pr}(S') = \left[ \begin{pmatrix} & \varpi^{-c} \\ -\varpi^{c+e-1} & \end{pmatrix} \right].$$

Follow from this

$$\sum_{t \in K^\times / F^\times \mathcal{O}_c^\times} \Psi(t) \chi(t) = 1 + \sum_{i=0}^{c-1} \Psi_i \sum_{t \in S_i} \chi(t) + \Psi' \sum_{t \in S'} \chi(t),$$

where  $\Psi_i$  (resp.  $\Psi'$ ) are the valuations of  $\Psi(t)$  on  $S_i$  (resp.  $S'$ ).

Assume the central character  $\omega$  is unramified, then we may take  $\omega = 1$ . If  $e = c = 1$ , we have

$$\sum_{t \in S_0} \chi(t) = -\chi(\tau) - 1, \quad \sum_{t \in S'} \chi(t) = \chi(\tau).$$

Otherwise

$$\sum_{t \in S_i} \chi(t) = \begin{cases} 0, & \text{if } c > 1 \text{ and } 0 \leq i \leq c-2, \\ -1, & \text{if } i = c-1, \end{cases} \quad \text{and} \quad \sum_{t \in S'} \chi(t) = 0.$$

Therefore,

$$\sum_{t \in K^\times / F^\times \mathcal{O}_c^\times} \Psi(t) \chi(t) = \begin{cases} 1 + (-\chi(\tau) - 1) \Psi_0 + \chi(\tau) \Psi', & \text{if } e = c = 1, \\ 1 - \Psi_{c-1}, & \text{otherwise.} \end{cases}$$

Note that if  $e = 1$ , then  $\begin{pmatrix} & \varpi^{-c} \\ -\varpi^c & \end{pmatrix}$  equals  $\begin{pmatrix} 1 & \varpi^{-c} \\ & 1 \end{pmatrix}$  in  $ZU_1(n) \backslash G/U_1(n)$  and as  $\omega = 1$ ,  $\Psi' = \Psi_0$ . We obtain

$$\sum_{t \in K^\times / F^\times \mathcal{O}_c^\times} \Psi(t) \chi(t) = 1 - \Psi_{c-1}$$

and reduce to evaluate  $\Psi_{c-1}$ . If  $n = 0$ , the matrix coefficient  $\Psi_{c-1}$  is expressed by the MacDonald polynomial. In particular, if the Satake parameter of  $\pi$  is  $(\alpha, \alpha^{-1})$ , then

$$1 - \Psi_{c-1} = \frac{(1 - \alpha^2 q^{-1})(1 - \alpha^{-2} q^{-1})}{1 + q^{-1}}.$$

If  $n = 1$ , then  $\pi = \text{sp}(2) \otimes \mu$  with  $\mu$  a unramified quadratic character on  $F^\times$ . By definition,

$$\begin{aligned} \Psi_{c-1} &= |\delta|^{1/2} L(1, \pi, \text{ad})^{-1} \int_{F^\times} W_0 \left[ \begin{pmatrix} a & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \varpi^{-1} \\ & 1 \end{pmatrix} \right] \overline{W_0 \left[ \begin{pmatrix} a & \\ & 1 \end{pmatrix} \right]} d^\times a \\ &= |\delta|^{3/2} L(1, \pi, \text{ad})^{-1} \int_{\varpi^{-n(\psi)} \mathcal{O}^\times} \psi(a \varpi^{-1}) |a|^2 d^\times a \\ &= |\delta|^{3/2} L(1, \pi, \text{ad})^{-1} (-q^{-1}) L(1, \pi, \text{ad}) |\delta|^{-3/2} = -q^{-1}. \end{aligned}$$

If  $n \geq 2$ , then

$$\Psi_{c-1} = |\delta|^{-1/2} \int_{\varpi^{-1-n(\psi)} \mathcal{O}^\times} \psi(x) d^\times x = -q^{-1} L(1, 1_F).$$

With these results, we obtain

$$\beta^0 = \frac{\text{Vol}(K^\times / F^\times)}{\#K^\times / F^\times \mathcal{O}_c^\times} \cdot \begin{cases} \frac{L(1, 1_F)}{L(1, \pi, \text{ad})(1 + q^{-1})}, & \text{if } n = 0; \\ 1 + q^{-1}, & \text{if } n = 1; \\ L(1, 1_F), & \text{if } n \geq 2. \end{cases}$$

Finally, we deal with the case  $\omega$  is ramified. As above, it is routine to check that  $\Psi_i$  with  $i < c-1$  and  $\Psi'$  are vanishing. Moreover,  $\Psi_{c-1} = 0$  if and only if  $\delta_\pi = 0$  and for  $\delta_\pi = 1$ ,

$$\Psi_{c-1} = -q^{-1} L(1, 1_F).$$

By the definition of  $\delta_\pi$ , if  $\delta_\pi = 1$  then  $c \geq 2$  and  $n(\omega) < n \leq c$ . Thus, for  $\delta_\pi = 1$

$$\begin{aligned} 0 &= \sum_{t \in 1 + \varpi^{c-1}\mathcal{O}_K/1 + \varpi^c\mathcal{O}_K} \chi(t) \\ &= \sum_{t \in 1 + \varpi^{c-1}\mathcal{O}_K/(1 + \varpi^{c-1}\mathcal{O})(1 + \varpi^c\mathcal{O}_K)} \chi(t) \sum_{a \in 1 + \varpi^{c-1}\mathcal{O}/1 + \varpi^c\mathcal{O}} \omega^{-1}(a) \\ &= q \sum_{b \in \mathfrak{p}^{c-1}/\mathfrak{p}^c} \chi(1 + b\tau), \end{aligned}$$

Therefore, if  $\delta_\pi = 1$ , then  $\sum_{t \in S_{c-1}} \chi(t) = -1$  and

$$\beta^0 = \frac{\text{Vol}(K^\times/F^\times)}{\#K^\times/F^\times\mathcal{O}_c^\times} \cdot \begin{cases} 1, & \text{if } \delta_\pi = 0; \\ L(1, 1_F), & \text{if } \delta_\pi = 1. \end{cases}$$

The proof of Proposition 3.12 is now complete.

We finish our discussions on  $\alpha(W_0)$ ,  $\beta(f)$  and  $\gamma$  by the following Lemmas 3.13 and 3.14.

**Lemma 3.13.** *Let  $F$  be nonarchimedean and  $f$  a nonzero element in  $V(\pi, \chi)$  then*

$$\alpha(W_0)\beta(f)\gamma|D|^{-1/2} = 2^{\delta(\Sigma_D)}L(1/2, \pi, \chi)^{-\delta(\Sigma)}L(1, \eta)^{2\delta(c_1)}q^{-c_1},$$

where these  $\delta \in \{0, 1\}$  defined by

- $\delta(\Sigma_D) = 1$  if and only if  $K$  is ramified,  $n > 0$  and  $c < n$ ;
- $\delta(\Sigma) = 1$  if and only if  $n > 0$ ,  $K$  is either ramified or  $c > 0$  and if  $n = 1$ , then  $c \geq n$ ;
- $\delta(c_1) = 1$  if and only if  $c_1 \neq 0$ .

*Proof.* We have computed  $\alpha(W_0)$  in Proposition 3.11 and  $\beta(f)$  in Proposition 3.12. When  $n > 0$ , by Lemma 3.5,  $\gamma = L(1, 1_F)(1 - e(R)q^{-1})$  and we reduce to compute  $e(R)$ :

- (i)  $e(R) = 1$  and  $\gamma = 1$  if  $K$  is split, or  $K$  is ramified,  $n = 1$  and  $B$  is split, or  $K$  is nonsplit and  $c \geq n$ ;
- (ii)  $e(R) = -1$  and  $\gamma = L(1, 1_F)(1 + q^{-1})$  if  $K$  is inert and  $c < n$ , or  $K$  is ramified,  $n = 1$ ,  $B$  is division and  $c = 0$ ;
- (iii)  $e(R) = 0$  and  $\gamma = L(1, 1_F)$  if  $K$  is ramified,  $n \geq 2$  and  $c < n$ .

□

For archimedean places, using Barnes' lemma, we have the following list for  $(W_0, W_0)$  (see [30] for the classification of unitary dual of  $\text{GL}_2(F)$ ):

- (1) Assume  $F = \mathbb{R}$ ,  $\sigma$  is the infinite dimensional subquotient of the induced representation  $\text{Ind}(\mu_1, \mu_2)$  where  $\mu_i(a) = |a|^{s_i} \text{sgn}(a)^{m_i}$  with  $s_i \in \mathbb{C}$  and  $m_i \in \{0, 1\}$ . Denote  $k = s_1 - s_2 + 1$ ,  $\mu = s_1 + s_2$ .
  - (a) If  $\sigma = \mathcal{D}_\mu(k)$  is the discrete series with  $k \geq 2$ , then  $(W_0, W_0)$  equals

$$2(4\pi)^{-k}\Gamma(k).$$

- (b) If  $\sigma = \pi(\mu_1, \mu_2)$  is a principal series, then  $(W_0, W_0)$  equals

$$\pi^{-1-m_1-m_2}\Gamma\left(\frac{1+2m_1}{2}\right)\Gamma\left(\frac{1+2m_2}{2}\right)B\left(\frac{k+m_1+m_2}{2}, \frac{2-k+m_1+m_2}{2}\right),$$

where  $B(x, y) := \Gamma(x)\Gamma(y)\Gamma(x+y)^{-1}$  is the beta function.

- (2) Assume  $F = \mathbb{C}$ ,  $\sigma = \pi(\mu_1, \mu_2)$  is a principal series with  $\mu_i(z) = |z|^{s_i}(z/\sqrt{|z|})^{m_i}$  and  $s_i \in \mathbb{C}$  and  $m_i \in \mathbb{Z}$ , then  $(W_0, W_0)$  equals

$$8(2\pi)^{-1-|m_1|-|m_2|}\Gamma(1+|m_1|)\Gamma(1+|m_2|)B\left(1+s_1-s_2+\frac{|m_1|+|m_2|}{2}, 1-s_1+s_2+\frac{|m_1|+|m_2|}{2}\right).$$

For a pair  $(\pi, \chi)$ , define

$$C(\pi, \chi) = \begin{cases} 2^{-1}\pi(W_0, W_0)^{-1}, & \text{if } K/F = \mathbb{C}/\mathbb{R}; \\ (W'_0, W'_0)(W_0, W_0)^{-1}, & \text{if } K = F^2. \end{cases}$$

In the split case,  $W'_0$  is the new vector of  $\pi \otimes \chi_1$  where  $K$  is embedded into  $M_2(F)$  diagonally and

$$\chi_1(a) = \chi\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right).$$

**Lemma 3.14.** *For  $F$  archimedean, let  $f$  be a nonzero vector in  $V(\pi, \chi)$ , then*

$$\alpha(W_0)\beta(f) = C(\pi, \chi)^{-1} \begin{cases} L(1/2, \pi, \chi)^{-1}, & \text{if } K/F = \mathbb{C}/\mathbb{R}; \\ 1, & \text{if } K = F^2. \end{cases}$$

*In particular, if  $\sigma = \mathcal{D}_\mu(k)$  is a discrete series with weight  $k$ , then*

$$C(\pi, \chi) = \begin{cases} 4^{k-1} \pi^{k+1} \Gamma(k)^{-1}, & \text{if } K = \mathbb{C}; \\ 1, & \text{if } K = \mathbb{R}^2. \end{cases}$$

*Proof.* By definition,

$$\alpha(W_0)\beta(f) = \frac{L(1, \eta)}{L(1, 1_F)} L(1/2, \pi, \chi)^{-1} (W_0, W_0) \beta^0,$$

with

$$\beta^0 = \int_{F^\times \backslash K^\times} \frac{(\pi(t)f, f)}{(f, f)} \chi(t) dt, \quad f \in V(\pi, \chi).$$

If  $K/F = \mathbb{C}/\mathbb{R}$ , then  $\beta^0 = \text{Vol}(K^\times/F^\times) = 2$ . If  $K$  is split, taking  $f = W'_0$ , then  $\beta^0 = L(1/2, \pi, \chi)(W'_0, W'_0)^{-1}$ . If  $\sigma = \mathcal{D}_\mu(k)$ , the value for  $(W_0, W_0)$  is in the above list (1a) and we note that if  $K = \mathbb{R}^2$ , then  $(W'_0, W'_0) = (W_0, W_0)$  as for any  $\chi_1, \pi \otimes \chi_1$  and  $\pi$  have the same weight.  $\square$

## REFERENCES

- [1] M. Bertolini and H. Darmon, *A rigid analytic Gross-Zagier formula and arithmetic applications*, Annals of Math 146(1997), 111-147.
- [2] M. Bertolini and H. Darmon, *Iwasawa's main conjecture for elliptic curves over anticyclotomic  $\mathbb{Z}_p$ -extensions*, Ann. of Math. 162 (2005), 1-64.
- [3] B.J. Birch, *Elliptic curves and modular functions*, Symposia Mathematica, Indam Rome 1968/1969, vol. 4, pp27-32. London: Academic Press (1970).
- [4] B.J. Birch and H.P.F. Swinnerton-Dyer, *Notes on elliptic curves (II)*, J. Reine Angew. Math, 218 (1965), 79-108.
- [5] L. Cai, J. Shu, and Y. Tian, *Cube Sum Problem and an Explicit Gross-Zagier Formula*, preprint.
- [6] H. Carayol, *Représentations cuspidales du groupe linéaire*, Ann.Sci. Ecole Norm. Sup. 17 (1984) 191-225.
- [7] W. Casselman, *On some results of Atkin and Lehner*, Math. Ann. 201 (1973), 301-314.
- [8] W. Casselman, *The Restriction of a representation of  $\text{GL}_2(k)$  to  $\text{GL}_2(\mathcal{O})$* , Math. Ann, 206(1973), 311-318.
- [9] J. Coates, Y. Li, Y. Tian, and S. Zhai, *Quadratic Twists of Elliptic Curves*, to appear in Proceedings of the London Mathematical Society.
- [10] N. Chen and H. Jacquet, *Positivity of quadratic base change  $L$ -functions*, Bull Soc. math. France 129 (1), 2001, p. 33-90.
- [11] T. Dokchitser and V. Dokchitser, *On the Birch-Swinnerton-Dyer quotient modulo squares*, Annals of Mathematics, 172 (2010), 567-596.
- [12] S. Dasgupta and J. Voight, *Heegner Points and Sylvester's Conjecture*, Arithmetic Geometry, Clay Mathematics Proceedings, 8. American Mathematical Society, Providence, RI. 2009.
- [13] B. Gross, *Heights and the special values of  $L$ -series*, Canadian Mathematical Society, Conference Proceedings, Volume 7 (1987).
- [14] B. Gross, *Local orders, root numbers, and modular curves*, Amer. J. Math., 110(1988), 1153-1182.
- [15] B. Gross and D. Prasad, *Test vectors for linear forms*, Math. Ann, 291(1991), 343-355.
- [16] B. Gross and D. Zagier, *Heegner points and derivatives of  $L$ -series*, Invent. Math. 84 (1986), no. 2, 225-320.
- [17] K. Heegner, *Diophantische analysis und modulfunktionen*, Math. Z. 56, 227-253 (1952).
- [18] T. Miyake, *Modular Forms*, Springer Monographs in Mathematics, 1989.
- [19] P. Monsky, *Mock Heegner Points and Congruent Numbers*, Math. Z. 204, 45-68 (1990).
- [20] C. Moeglin and J.-L. Waldspurger, *Spectral decomposition and Eisenstein series*, Cambridge Tracts in Mathematics, vol. 113, Cambridge University Press, Cambridge, 1995.
- [21] J. Nekovář, *The Euler system method for CM points on Shimura curves*, In: L-functions and Galois representations, (Durham, July 2004), LMS Lecture Note Series 320, Cambridge Univ. Press, 2007, pp. 471 - 547.
- [22] B. Perrin-Riou, *Points de Heegner et dérivées de fonctions  $L$   $p$ -adiques*, Invent. Math. 89 (1987), no. 3, pp. 455-510.
- [23] A. Popa, *Whittaker newforms for archimedean representations of  $\text{GL}(2)$* , J. Number Theory 128 (2008), no. 6, 1637-1645.
- [24] D. Prasad, *Some applications of seesaw duality to branching laws*, Math. Ann. 304, 1-20 (1996).
- [25] D. Prasad, *Invariant linear forms for representations of  $\text{GL}(2)$  over a local field*, Amer. J. Math 114 (1992), 1317-1363.
- [26] R. Schmidt, *Some remarks on local newforms for  $\text{GL}(2)$* , J. Ramanujan Math. Soc. 17 (2002), 115-147.
- [27] H. Saito, *On Tunnell's formula for characters of  $\text{GL}(2)$* , Compositio Math. 85 (1993), 99-108.
- [28] P. Satgé, *Un analogue du calcul de Heegner*, Invent. Math. 87 (1987), 425-439.
- [29] C. Skinner and E. Urban, *The Iwasawa Main Conjectures for  $\text{GL}_2$* , Invent. math (2014) 195: 1-277.
- [30] M. Tadic,  *$\text{GL}(n, \mathbb{C})$  and  $\text{GL}(n, \mathbb{R})$  in Automorphic Forms and L-functions II, Local Aspects*, Contemporary Math. 489 (2009), 285-313.
- [31] J. Tate, *Number Theoretic Background in Automorphic Forms, Representations and  $L$ -functions*, Proc. Symp. in Pure Math. XXXIII Part 2(1979), 3-26.

- [32] Y. Tian, *Congruent Numbers with many prime factors*, PNAS, Vol 109, no. 52. 21256-21258.
- [33] Y. Tian, *Congruent Numbers and Heegner Points*, Cambridge J. Math. Vol. 2.1. 117-161, 2014.
- [34] Y. Tian, X. Yuan, and S. Zhang, *Genus Periods, Genus Points and Congruent Number Problem*, preprint.
- [35] J. Tunnell, *On the local Langlands conjecture for  $GL(2)$* , Invent. Math. 46 (1978), 179-200.
- [36] J. Tunnell, *Local  $\epsilon$ -factors and characters of  $GL(2)$* , Amer. J. Math. 105 (1983), 1277- 1307.
- [37] M-F. Vignéras, *Arithmétique des Algèbres de Quaternions*, Lecture Notes in Mathematics, 800. Springer, Berlin, 1980. vii+169 pp.
- [38] J.-L. Waldspurger, *Sur les valeurs de certaines fonctions  $L$  automorphes en leur centre de symétrie*, Compositio Math. 54 (1985), no. 2, 173-242.
- [39] L. Washington, *Introduction to cyclotomic Fields*, 2nd ed., GTM 83, Springer, 1997.
- [40] X.Yuan, S. Zhang, and W. Zhang, *The Gross-Zagier Formula on Shimura Curves*, Annals of Mathematics Studies Number 184, 2013.
- [41] C. Yu, *Variants of mass formulas for definite division algebras*, preprint.
- [42] S. Zhang, *Height of Heegner points on Shimura curves*, Annals of Mathematics(2), 153 (2001), no. 1, 27-147.
- [43] S. Zhang, *Gross-Zagier formula for  $GL_2$* , Asian J. Math., 5(2001), no 2, 183-290.
- [44] S. Zhang, *Gross-Zagier formula for  $GL(2)$  II*, Heegner points and Rankin L-series, 191-254, Math. Sci. Res. Inst. Publ. 49, Cambridge Univ. Press, Cambridge, 2004.

LI CAI: MATHEMATICAL SCIENCES CENTER, TSINGHUA UNIVERSITY, BEIJING 100084  
*E-mail address:* `lcail@math.tsinghua.edu.cn`

JIE SHU: ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCE, MORNINGSIDE CENTER OF MATHEMATICS, CHINESE ACADEMY OF SCIENCES, BEIJING 100190  
*E-mail address:* `shujie09@mails.gucas.ac.cn`

YE TIAN: ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCE, MORNINGSIDE CENTER OF MATHEMATICS, CHINESE ACADEMY OF SCIENCES, BEIJING 100190  
*E-mail address:* `ytian@math.ac.cn`